Summer Assignment for Honors Calculus

Problems below are from Anton Calculus Book, 7th edition, pages attached.

Appendix A, Read examples 4 and 5 carefully, p. A10: 23-45 odd, 46

Appendix B, Read example 4, p. A 15: 17-35 odd

Appendix C, p A 25-27: 5, 7, 11, 19, 23, 33e, 41, 43

Appendix D, p. A 34-36: 9, 29, 33, 37, 41, 47, 48, 51, 59, 69, 77, 81

Problems below are from Anton Calculus Book , 9th edition.
You have a PDF of the book.
Read each section carefully before working out problems.

Appendix B. Trigonometry Review: p. A23-A25: 1-53 odd

Appendix C: Polynomial equations: p. A34-35: 1-23 odd

Directions

This summer complete all problems above.

If you have difficulty with a problem, make sure you have read the examples.

You should be able to do the problems without a calculator, but you should check your answers on the calculator. Also, odd answers are in the back of the book.

APPENDIX A

Real Numbers, Intervals, and Inequalities

REAL NUMBERS

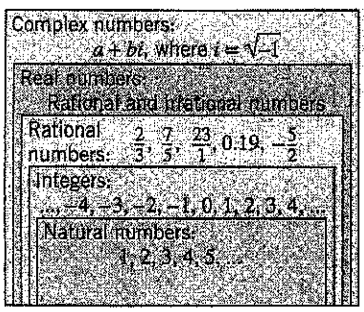

Figure A.1

Figure A.1 describes the various categories of numbers that we will encounter in this text. The simplest numbers are the ***natural numbers***

$$1,\ 2,\ 3,\ 4,\ 5, \ldots$$

These are a subset of the ***integers***

$$\ldots,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4, \ldots$$

and these in turn are a subset of the ***rational numbers***, which are the numbers formed by taking ratios of integers (avoiding division by 0). Some examples are

$$\tfrac{2}{3},\quad \tfrac{7}{5},\quad 23 = \tfrac{23}{1},\quad 0.19 = \tfrac{19}{100},\quad -\tfrac{5}{2} = \tfrac{-5}{2} = \tfrac{5}{-2}$$

The early Greeks believed that every measurable quantity had to be a rational number. However, this idea was overturned in the fifth century B.C. by Hippasus of Metapontum[*] who demonstrated the existence of ***irrational numbers***, that is, numbers that cannot be expressed as the ratio of two integers. Using geometric methods, he showed that the length of the hypotenuse of the triangle in Figure A.2 could not be expressed as a ratio of integers, thereby proving that $\sqrt{2}$ is an irrational number. Some other examples of irrational numbers are

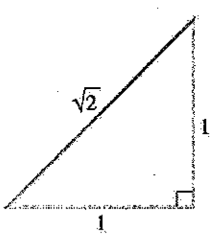

Figure A.2

$$\sqrt{3},\quad \sqrt{5},\quad 1+\sqrt{2},\quad \sqrt[3]{7},\quad \pi,\quad \cos 19^\circ$$

The rational and irrational numbers together comprise what is called the ***real number system***, and both the rational and irrational numbers are called ***real numbers***.

COMPLEX NUMBERS

Because the square of a real number cannot be negative, the equation

$$x^2 = -1$$

has no solutions in the real number system. In the eighteenth century mathematicians remedied this problem by inventing a new number, which they denoted by

$$i = \sqrt{-1}$$

and which they defined to have the property $i^2 = -1$. This, in turn, led to the development

[*] HIPPASUS OF METAPONTUM (circa 500 B.C.). A Greek Pythagorean philosopher. According to legend, Hippasus made his discovery at sea and was thrown overboard by fanatic Pythagoreans because his result contradicted their doctrine. The discovery of Hippasus is one of the most fundamental in the entire history of science.

of the ***complex numbers***, which are numbers of the form

$$a + bi$$

where a and b are real numbers. Some examples are

$2 + 3i$	$3 - 4i$	$6i$	$\frac{2}{3}$
$[a = 2, b = 3]$	$[a = 3, b = -4]$	$[a = 0, b = 6]$	$[a = \frac{2}{3}, b = 0]$

Observe that every real number a is also a complex number because it can be written as

$$a = a + 0i$$

Thus, the real numbers are a subset of the complex numbers. Although we will be concerned primarily with real numbers in this text, complex numbers will arise in the course of solving equations. For example, the solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

which are given by the ***quadratic formula***

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are not real if the quantity $b^2 - 4ac$ is negative.

DIVISION BY ZERO

Division by zero is not allowed in numerical computations because it leads to mathematical inconsistencies. For example, if $1/0$ were assigned some numerical value, say p, then it would follow that $0 \cdot p = 1$, which is incorrect.

DECIMAL REPRESENTATION OF REAL NUMBERS

Rational and irrational numbers can be distinguished by their decimal representations. Rational numbers have decimals that are ***repeating***, by which we mean that at some point in the decimal some fixed block of numbers begins to repeat indefinitely. For example,

$$\underbrace{\tfrac{4}{3} = 1.333\ldots}_{3 \text{ repeats}}, \quad \underbrace{\tfrac{3}{11} = .272727\ldots}_{27 \text{ repeats}}, \quad \underbrace{\tfrac{1}{2} = .50000\ldots}_{0 \text{ repeats}}, \quad \underbrace{\tfrac{5}{7} = .714285714285714285\ldots}_{714285 \text{ repeats}}$$

Decimals in which zero repeats from some point on are called ***terminating decimals***. For brevity, it is usual to omit the repetitive zeros in terminating decimals and for other repeating decimals to write the repeating digits only once but with a bar over them to indicate the repetition. For example,

$$\tfrac{1}{2} = .5, \quad \tfrac{12}{4} = 3, \quad \tfrac{8}{25} = .32, \quad \tfrac{4}{3} = 1.\overline{3}, \quad \tfrac{3}{11} = .\overline{27}, \quad \tfrac{5}{7} = .\overline{714285}$$

Irrational numbers have nonrepeating decimals, so we can be certain that the decimals

$$\sqrt{2} = 1.4142135623730 95\ldots \quad \text{and} \quad \pi = 3.141592653589793\ldots$$

do not repeat from some point on. Moreover, if we stop the decimal expansion of an irrational number at some point, we get only an approximation to the number, never an exact value. For example, even if we compute π to 1000 decimal places, as in Figure A.3, we still have only an approximation.

```
3.141592653589793238462643383279502884197169
39937510582097494459230781640628620899862803
48253421170679821480865132823066470938446095
50582231725359408128481117450284102701938521
10555964462294895493038196442881097566593344
61284756482337867831652712019091456485669234
60348610454326648213393607260249141273724587
00660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415
11609433057270365759591953092186117381932611
79310511854807446237996274956735188575272489
12279381830119491298336733624406566430860213
94946395224737190702179860943702770539217176
29317675238467481846766940513200056812714526
35608277857713427577896091736371787214684409
01224953430146549585371050792279689258923542
01995611212902196086403441815981362977477130
99605187072113499999983729780499510597317328
16096318595024459455346908302642522308253344
68503526193118817101000313783875288658753320
83814206171776691473035982534904287554687311
59562863882353787593751957781857780532171226
8066130019278766111959092164201989
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Figure A.3

REMARK. Beginning mathematics students are sometimes taught to approximate π by $\frac{22}{7}$. Keep in mind, however, that this is only an approximation, since

$$\tfrac{22}{7} = 3.\overline{142857}$$

is a rational number whose decimal representation begins to differ from π in the third decimal place.

COORDINATE LINES

In 1637 René Descartes* published a philosophical work called *Discourse on the Method of Rightly Conducting the Reason*. In the back of that book was an appendix that the British philosopher John Stuart Mill described as "the greatest single step ever made in the progress of the exact sciences." In that appendix René Descartes linked together algebra and geometry, thereby creating a new subject called ***analytic geometry***; it gave a way of describing algebraic formulas by geometric curves and, conversely, geometric curves by algebraic formulas.

The key step in analytic geometry is to establish a correspondence between real numbers and points on a line. To do this, choose any point on the line as a reference point, and call it the ***origin***; and then arbitrarily choose one of the two directions along the line to be the ***positive direction***, and let the other be the ***negative direction***. It is usual to mark the positive direction with an arrowhead, as in Figure A.4, and to take the positive direction to the right when the line is horizontal. Next, choose a convenient unit of measure, and represent each positive number r by the point that is r units from the origin in the positive direction, each negative number $-r$ by the point that is r units from the origin in the negative direction from the origin, and 0 by the origin itself (Figure A.5). The number associated with a point P is called the ***coordinate*** of P, and the line is called a ***coordinate line***, a ***real number line***, or a ***real line***.

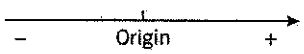


Figure A.4

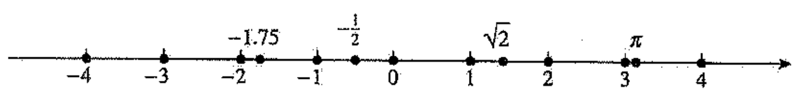


Figure A.5

INEQUALITY NOTATION

The real numbers can be ordered by size as follows: If $b - a$ is positive, then we write either $a < b$ (read "a is less than b") or $b > a$ (read "b is greater than a"). We write $a \leq b$ to mean $a < b$ or $a = b$, and we write $a < b < c$ to mean that $a < b$ and $b < c$. As one traverses a coordinate line in the positive direction, the real numbers increase in size, so on a horizontal coordinate line the inequality $a < b$ implies that a is to the left of b, and the inequalities $a < b < c$ imply that a is to the left of c, and b lies between a and c. The meanings of such symbols as

$$a \leq b < c, \quad a \leq b \leq c, \quad \text{and} \quad a < b < c < d$$

should be clear. For example, you should be able to confirm that all of the following are true statements:

$$3 < 8, \quad -7 < 1.5, \quad -12 \leq -\pi, \quad 5 \leq 5, \quad 0 \leq 2 \leq 4,$$
$$8 \geq 3, \quad 1.5 > -7, \quad -\pi > -12, \quad 5 \geq 5, \quad 3 > 0 > -1 > -3$$

REVIEW OF SETS

In the following discussion we will be concerned with certain sets of real numbers, so it will be helpful to review the basic ideas about sets. Recall that a ***set*** is a collection of objects, called ***elements*** or ***members*** of the set. In this text we will be concerned primarily with sets whose members are numbers or points that lie on a line, a plane, or in three-dimensional space. We will denote sets by capital letters and elements by lowercase letters. To indicate that a is a member of the set A we will write $a \in A$ (read "a belongs to A"), and to indicate

*RENÉ DESCARTES (1596–1650). Descartes, a French aristocrat, was the son of a government official. He graduated from the University of Poitiers with a law degree at age 20. After a brief probe into the pleasures of Paris he became a military engineer, first for the Dutch Prince of Nassau and then for the German Duke of Bavaria. It was during his service as a soldier that Descartes began to pursue mathematics seriously and develop his analytic geometry. After the wars, he returned to Paris where he stalked the city as an eccentric, wearing a sword in his belt and a plumed hat. He lived in leisure, seldom arose before 11 A.M., and dabbled in the study of human physiology, philosophy, glaciers, meteors, and rainbows. He eventually moved to Holland, where he published his *Discourse on the Method*, and finally to Sweden where he died while serving as tutor to Queen Christina. Descartes is regarded as a genius of the first magnitude. In addition to major contributions in mathematics and philosophy, he is considered, along with William Harvey, to be a founder of modern physiology.

that a is not a member of the set A we will write $a \notin A$ (read "a does not belong to A"). For example, if A is the set of positive integers, then $5 \in A$, but $-5 \notin A$. Sometimes sets arise that have no members (e.g., the set of odd integers that are divisible by 2). A set with no members is called an ***empty set*** or a ***null set*** and is denoted by the symbol $\varnothing$.

Some sets can be described by listing their members between braces. The order in which the members are listed does not matter, so, for example, the set A of positive integers that are less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\} \quad \text{or} \quad A = \{2, 3, 1, 5, 4\}$$

We can also write A in *set-builder notation* as

$$A = \{x : x \text{ is an integer and } 0 < x < 6\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 6$." In general, to express a set S in set-builder notation we write $S = \{x : \underline{\quad\quad}\}$ in which the line is replaced by a property that identifies exactly those elements in the set S.

If every member of a set A is also a member of a set B, then we say that A is a ***subset*** of B and write $A \subseteq B$. For example, if A is the set of positive integers and B is the set of all integers, then $A \subseteq B$. If two sets A and B have the same members (i.e., $A \subseteq B$ and $B \subseteq A$), then we say that A and B are ***equal*** and write $A = B$.

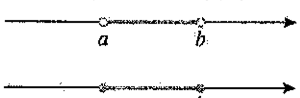
INTERVALS

In calculus we will be concerned with sets of real numbers, called ***intervals***, that correspond to line segments on a coordinate line. For example, if $a < b$, then the ***open interval*** from a to b, denoted by (a, b), is the line segment extending from a to b, *excluding* the endpoints; and the ***closed interval*** from a to b, denoted by $[a, b]$, is the line segment extending from a to b, *including* the endpoints (Figure A.6). These sets can be expressed in set-builder notation as

$$(a, b) = \{x : a < x < b\} \qquad \text{The open interval from } a \text{ to } b$$

$$[a, b] = \{x : a \le x \le b\} \qquad \text{The closed interval from } a \text{ to } b$$

The open interval (a, b)



The closed interval $[a, b]$

Figure A.6

REMARK. Observe that in this notation and in the corresponding Figure A.6, parentheses and open dots mark endpoints that are excluded from the interval, whereas brackets and closed dots mark endpoints that are included in the interval. Observe also that in set-builder notation for the intervals, it is understood that x is a real number, even though it is not stated explicitly.

As shown in Table 1, an interval can include one endpoint and not the other; such intervals are called ***half-open*** (or sometimes ***half-closed***). Moreover, the table also shows that it is possible for an interval to extend indefinitely in one or both directions. To indicate that an interval extends indefinitely in the positive direction we write $+\infty$ (read "positive infinity") in place of a right endpoint, and to indicate that an interval extends indefinitely in the negative direction we write $-\infty$ (read "negative infinity") in place of a left endpoint. Intervals that extend between two real numbers are called ***finite intervals***, whereas intervals that extend indefinitely in one or both directions are called ***infinite intervals***.

REMARK. By convention, infinite intervals of the form $[a, +\infty)$ or $(-\infty, b]$ are considered to be closed because they contain their endpoint, and intervals of the form $(a, +\infty)$ and $(-\infty, b)$ are considered to be open because they do not include their endpoint. The interval $(-\infty, +\infty)$, which is the set of all real numbers, has no endpoints and can be regarded as either open or closed, as convenient. This set is often denoted by the special symbol $\mathbb{R}$. To distinguish verbally between the open interval $(0, +\infty) = \{x : x > 0\}$ and the closed interval $[0, +\infty) = \{x : x \ge 0\}$, we will call x ***positive*** if $x > 0$ and ***nonnegative*** if $x \ge 0$. Thus, a positive number must be nonnegative, but a nonnegative number need not be positive, since it might possibly be 0.

Table 1

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a, b)	$\{x : a < x < b\}$		Finite; open
$[a, b]$	$\{x : a \le x \le b\}$		Finite; closed
$[a, b)$	$\{x : a \le x < b\}$		Finite; half-open
$(a, b]$	$\{x : a < x \le b\}$		Finite; half-open
$(-\infty, b]$	$\{x : x \le b\}$		Infinite; closed
$(-\infty, b)$	$\{x : x < b\}$		Infinite; open
$[a, +\infty)$	$\{x : x \ge a\}$		Infinite; closed
$(a, +\infty)$	$\{x : x > a\}$		Infinite; open
$(-\infty, +\infty)$	$\mathbb{R}$		Infinite; open and closed

UNIONS AND INTERSECTIONS OF INTERVALS

If A and B are sets, then the ***union*** of A and B (denoted by $A \cup B$) is the set whose members belong to A or B (or both), and the ***intersection*** of A and B (denoted by $A \cap B$) is the set whose members belong to both A and B. For example,

$$\{x : 0 < x < 5\} \cup \{x : 1 < x < 7\} = \{x : 0 < x < 7\}$$
$$\{x : x < 1\} \cap \{x : x \ge 0\} = \{x : 0 \le x < 1\}$$
$$\{x : x < 0\} \cap \{x : x > 0\} = \varnothing$$

or in interval notation,

$$(0, 5) \cup (1, 7) = (0, 7)$$
$$(-\infty, 1) \cap [0, +\infty) = [0, 1)$$
$$(-\infty, 0) \cap (0, +\infty) = \varnothing$$

ALGEBRAIC PROPERTIES OF INEQUALITIES

The following algebraic properties of inequalities will be used frequently in this text. We omit the proofs.

A.1 THEOREM (***Properties of Inequalities***). *Let a, b, c, and d be real numbers.*
(*a*) *If $a < b$ and $b < c$, then $a < c$.*
(*b*) *If $a < b$, then $a + c < b + c$ and $a - c < b - c$.*
(*c*) *If $a < b$, then $ac < bc$ when c is positive and $ac > bc$ when c is negative.*
(*d*) *If $a < b$ and $c < d$, then $a + c < b + d$.*
(*e*) *If a and b are both positive or both negative and $a < b$, then $1/a > 1/b$.*

If we call the direction of an inequality its *sense*, then these properties can be paraphrased as follows:

(*b*) *The sense of an inequality is unchanged if the same number is added to or subtracted from both sides.*

(*c*) *The sense of an inequality is unchanged if both sides are multiplied by the same positive number, but the sense is reversed if both sides are multiplied by the same negative number.*

(*d*) *Inequalities with the same sense can be added.*

(*e*) *If both sides of an inequality have the same sign, then the sense of the inequality is reversed by taking the reciprocal of each side.*

REMARK. These properties remain true if the symbols $<$ and $>$ are replaced by $\leq$ and $\geq$ in Theorem A.1.

Example 1

STARTING INEQUALITY	OPERATION	RESULTING INEQUALITY
$-2 < 6$	Add 7 to both sides.	$5 < 13$
$-2 < 6$	Subtract 8 from both sides.	$-10 < -2$
$-2 < 6$	Multiply both sides by 3.	$-6 < 18$
$-2 < 6$	Multiply both sides by -3.	$6 > -18$
$3 < 7$	Multiply both sides by 4.	$12 < 28$
$3 < 7$	Multiply both sides by -4.	$-12 > -28$
$3 < 7$	Take reciprocals of both sides.	$\frac{1}{3} > \frac{1}{7}$
$-8 < -6$	Take reciprocals of both sides.	$-\frac{1}{8} > -\frac{1}{6}$
$4 < 5, -7 < 8$	Add corresponding sides.	$-3 < 13$

◀

SOLVING INEQUALITIES

A ***solution*** of an inequality in an unknown x is a value for x that makes the inequality a true statement. For example, $x = 1$ is a solution of the inequality $x < 5$, but $x = 7$ is not. The set of all solutions of an inequality is called its ***solution set***. It can be shown that if one does not multiply both sides of an inequality by zero or an expression involving an unknown, then the operations in Theorem A.1 will not change the solution set of the inequality. The process of finding the solution set of an inequality is called ***solving*** the inequality.

Example 2 Solve $3 + 7x \leq 2x - 9$.

Solution. We will use the operations of Theorem A.1 to isolate x on one side of the inequality.

$3 + 7x \leq 2x - 9$	Given.
$7x \leq 2x - 12$	We subtracted 3 from both sides.
$5x \leq -12$	We subtracted $2x$ from both sides.
$x \leq -\frac{12}{5}$	We multiplied both sides by $\frac{1}{5}$.

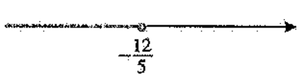


Figure A.7

Because we have not multiplied by any expressions involving the unknown x, the last inequality has the same solution set as the first. Thus, the solution set is the interval $\left(-\infty, -\frac{12}{5}\right]$ shown in Figure A.7. ◀

Example 3 Solve $7 \leq 2 - 5x < 9$.

Solution. The given inequality is actually a combination of the two inequalities

$$7 \leq 2 - 5x \quad \text{and} \quad 2 - 5x < 9$$

We could solve the two inequalities separately, then determine the values of x that satisfy both by taking the intersection of the two solution sets. However, it is possible to work with the combined inequalities in this problem:

$$7 \le 2 - 5x < 9 \qquad \boxed{\text{Given.}}$$

$$5 \le -5x < 7 \qquad \boxed{\text{We subtracted 2 from each member.}}$$

$$-1 \ge x > -\frac{7}{5} \qquad \boxed{\text{We multiplied by } -\tfrac{1}{5} \text{ and reversed the sense of the inequalities.}}$$

$$-\frac{7}{5} < x \le -1 \qquad \boxed{\text{For clarity, we rewrote the inequalities with the smaller number on the left.}}$$

Figure A.8

Thus, the solution set is the interval $\left(-\frac{7}{5}, -1\right]$ shown in Figure A.8. ◀

Example 4 Solve $x^2 - 3x > 10$.

Solution. By subtracting 10 from both sides, the inequality can be rewritten as

$$x^2 - 3x - 10 > 0$$

Factoring the left side yields

$$(x+2)(x-5) > 0$$

The values of x for which $x + 2 = 0$ or $x - 5 = 0$ are $x = -2$ and $x = 5$. These values divide the coordinate line into three open intervals,

$$(-\infty, -2), \quad (-2, 5), \quad (5, +\infty)$$

on each of which the product $(x+2)(x-5)$ has constant sign. To determine those signs we will choose an *arbitrary* number in each interval at which we will determine the sign; these are called ***test values***. As shown in Figure A.9, we will use -3, 0, and 6 as our test values. The results can be organized as follows:

INTERVAL	TEST VALUE	SIGN OF $(x+2)(x-5)$ AT THE TEST VALUE
$(-\infty, -2)$	-3	$(-)(-) = +$
$(-2, 5)$	0	$(+)(-) = -$
$(5, +\infty)$	6	$(+)(+) = +$

The pattern of signs in the intervals is shown on the number line in the middle of Figure A.9. We deduce that the solution set is $(-\infty, -2) \cup (5, +\infty)$, which is shown at the bottom of Figure A.9. ◀

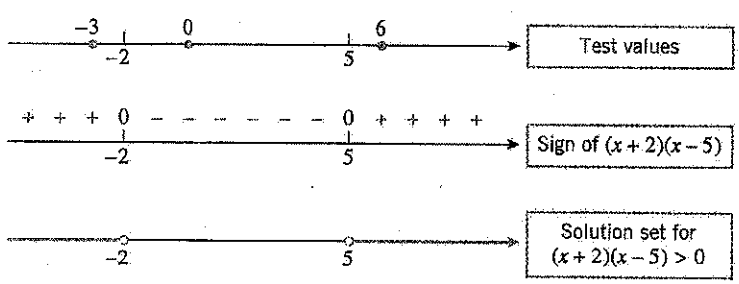


Figure A.9

Example 5 Solve $\dfrac{2x-5}{x-2} < 1$.

Solution. We could start by multiplying both sides by $x - 2$ to eliminate the fraction. However, this would require us to consider the cases $x - 2 > 0$ and $x - 2 < 0$ separately because the sense of the inequality would be reversed in the second case, but not the first. The following approach is simpler:

$$\frac{2x-5}{x-2} < 1 \quad \text{Given.}$$

$$\frac{2x-5}{x-2} - 1 < 0 \quad \text{We subtracted 1 from both sides to obtain a 0 on the right.}$$

$$\frac{(2x-5)-(x-2)}{x-2} < 0 \quad \text{We combined terms.}$$

$$\frac{x-3}{x-2} < 0 \quad \text{We simplified.}$$

The quantity $x - 3$ is zero if $x = 3$, and the quantity $x - 2$ is zero if $x = 2$. These values divide the coordinate line into three open intervals,

$$(-\infty, 2), \quad (2, 3), \quad (3, +\infty)$$

on each of which the quotient $(x-3)/(x-2)$ has constant sign. Using 0, 2.5, and 4 as test values (Figure A.10), we obtain the following results:

INTERVAL	TEST VALUE	SIGN OF $(x-3)/(x-2)$ AT THE TEST VALUE
$(-\infty, 2)$	0	$(-)/(-) = +$
$(2, 3)$	2.5	$(-)/(+) = -$
$(3, +\infty)$	4	$(+)/(+) = +$

The signs of the quotient are shown in the middle of Figure A.10. From the figure we see that the solution set consists of all real values of x such that $2 < x < 3$. This is the interval $(2, 3)$ shown at the bottom of Figure A.10. ◀

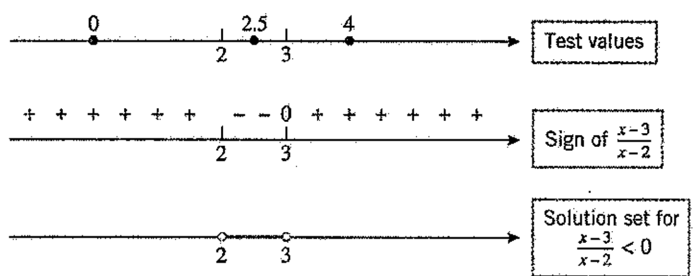


Figure A.10

Exercise Set A

1. Among the terms *integer*, *rational*, and *irrational*, which ones apply to the given number?
   (a) $-\frac{3}{4}$ (b) 0 (c) $\frac{24}{8}$
   (d) 0.25 (e) $-\sqrt{16}$ (f) $2^{1/2}$
   (g) 0.020202... (h) 7.000...

2. Which of the terms *integer*, *rational*, and *irrational* apply to the given number?
   (a) 0.31311311131111... (b) 0.729999...
   (c) 0.376237623762... (d) $17\frac{4}{5}$

3. The repeating decimal 0.137137137... can be expressed as a ratio of integers by writing

$$x = 0.137137137\ldots$$
$$1000x = 137.137137137\ldots$$

   and subtracting to obtain $999x = 137$ or $x = \frac{137}{999}$. Use this idea, where needed, to express the following decimals as ratios of integers.
   (a) 0.123123123... (b) 12.7777...
   (c) 38.07818181... (d) 0.4296000...

4. Show that the repeating decimal 0.99999... represents the number 1. Since 1.000... is also a decimal representation of 1, this problem shows that a real number can have two different decimal representations. [*Hint:* Use the technique of Exercise 3.]

5. The Rhind Papyrus, which is a fragment of Egyptian mathematical writing from about 1650 B.C., is one of the oldest known examples of written mathematics. It is stated in the papyrus that the area A of a circle is related to its diameter D by

$$A = \left(\tfrac{8}{9}D\right)^2$$

(a) What approximation to π were the Egyptians using?
(b) Use a calculating utility to determine if this approximation is better or worse than the approximation $\frac{22}{7}$.

6. The following are all famous approximations to π:

$\dfrac{333}{106}$ Adrian Athoniszoon, c. 1583

$\dfrac{355}{113}$ Tsu Chung-Chi and others

$\dfrac{63}{25}\left(\dfrac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$ Ramanujan

$\dfrac{22}{7}$ Archimedes

$\dfrac{223}{71}$ Archimedes

(a) Use a calculating utility to order these approximations according to size.
(b) Which of these approximations is closest to but larger than π?
(c) Which of these approximations is closest to but smaller than π?
(d) Which of these approximations is most accurate?

7. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a and b. The first line is already completed as an illustration.

a	b	$a<b$	$a \le b$	$a>b$	$a \ge b$	$a=b$
1	6	✓	✓			
6	1					
−3	5					
5	−3					
−4	−4					
0.25	$\frac{1}{3}$					
$-\frac{1}{4}$	$-\frac{3}{4}$					

Table Ex-7

8. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a, b, and c.

a	b	c	$a<b<c$	$a \le b \le c$	$a<b \le c$	$a \le b<c$
−1	0	2				
2	4	−3				
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$				
−5	−5	−5				
0.75	1.25	1.25				

Table Ex-8

9. Which of the following are always correct if $a \le b$?
(a) $a-3 \le b-3$ (b) $-a \le -b$ 4&5
(c) $3-a \le 3-b$ (d) $6a \le 6b$
(e) $a^2 \le ab$ (f) $a^3 \le a^2b$ 5&5

10. Which of the following are always correct if $a \le b$ and $c \le d$?
(a) $a+2c \le b+2d$ (b) $a-2c \le b-2d$
(c) $a-2c \ge b-2d$

11. For what values of a are the following inequalities valid?
(a) $a \le a$ (b) $a < a$

12. If $a \le b$ and $b \le a$, what can you say about a and b?

13. (a) If $a < b$ is true, does it follow that $a \le b$ must also be true?
(b) If $a \le b$ is true, does it follow that $a < b$ must also be true?

14. In each part, list the elements in the set.
(a) $\{x : x^2 - 5x = 0\}$
(b) $\{x : x$ is an integer satisfying $-2 < x < 3\}$

15. In each part, express the set in the notation $\{x : \_\_\_\_\_\_\}$.
(a) $\{1, 3, 5, 7, 9, \ldots\}$
(b) the set of even integers
(c) the set of irrational numbers
(d) $\{7, 8, 9, 10\}$

16. Let $A = \{1, 2, 3\}$. Which of the following sets are equal to A?
(a) $\{0, 1, 2, 3\}$ (b) $\{3, 2, 1\}$
(c) $\{x : (x-3)(x^2-3x+2) = 0\}$

17. In the accompanying figure, let

$S =$ the set of points inside the square
$T =$ the set of points inside the triangle
$C =$ the set of points inside the circle

and let a, b, and c be the points shown. Answer the following as true or false.

(a) $T \subseteq C$ (b) $T \subseteq S$
(c) $a \notin T$ (d) $a \notin S$
(e) $b \in T$ and $b \in C$ (f) $a \in C$ or $a \in T$
(g) $c \in T$ and $c \notin C$

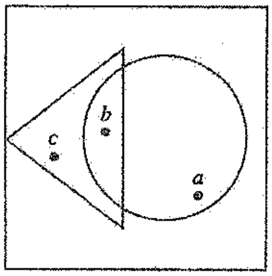


Figure Ex-17

18. List all subsets of
(a) $\{a_1, a_2, a_3\}$ (b) $\varnothing$.

19. In each part, sketch on a coordinate line all values of x that satisfy the stated condition.
(a) $x \le 4$ (b) $x \ge -3$ (c) $-1 \le x \le 7$
(d) $x^2 = 9$ (e) $x^2 \le 9$ (f) $x^2 \ge 9$

20. In parts (a)–(d), sketch on a coordinate line all values of x, if any, that satisfy the stated conditions.
(a) $x > 4$ and $x \le 8$
(b) $x \le 2$ or $x \ge 5$
(c) $x > -2$ and $x \ge 3$
(d) $x \le 5$ and $x > 7$

21. Express in interval notation.
(a) $\{x : x^2 \le 4\}$ (b) $\{x : x^2 > 4\}$

22. In each part, sketch the set on a coordinate line.
(a) $[-3, 2] \cup [1, 4]$ (b) $[4, 6] \cup [8, 11]$
(c) $(-4, 0) \cup (-5, 1)$ (d) $[2, 4) \cup (4, 7)$
(e) $(-2, 4) \cap (0, 5]$ (f) $[1, 2.3) \cup (1.4, \sqrt{2})$
(g) $(-\infty, -1) \cup (-3, +\infty)$ (h) $(-\infty, 5) \cap [0, +\infty)$

In Exercises 23–44, solve the inequality and sketch the solution on a coordinate line.

23. $3x - 2 < 8$ **24.** $\frac{1}{5}x + 6 \ge 14$
25. $4 + 5x \le 3x - 7$ **26.** $2x - 1 > 11x + 9$
27. $3 \le 4 - 2x < 7$ **28.** $-2 \ge 3 - 8x \ge -11$
29. $\dfrac{x}{x-3} < 4$ **30.** $\dfrac{x}{8-x} \ge -2$
31. $\dfrac{3x+1}{x-2} < 1$ **32.** $\dfrac{\frac{1}{2}x - 3}{4+x} > 1$
33. $\dfrac{4}{2-x} \le 1$ **34.** $\dfrac{3}{x-5} \le 2$
35. $x^2 > 9$ **36.** $x^2 \le 5$
37. $(x-4)(x+2) > 0$ **38.** $(x-3)(x+4) < 0$
39. $x^2 - 9x + 20 \le 0$ **40.** $2 - 3x + x^2 \ge 0$
41. $\dfrac{2}{x} < \dfrac{3}{x-4}$ **42.** $\dfrac{1}{x+1} \ge \dfrac{3}{x-2}$
43. $x^3 - x^2 - x - 2 > 0$ **44.** $x^3 - 3x + 2 \le 0$

In Exercises 45 and 46, find all values of x for which the given expression yields a real number.

45. $\sqrt{x^2 + x - 6}$ **46.** $\sqrt{\dfrac{x+2}{x-1}}$

47. Fahrenheit and Celsius temperatures are related by the formula $C = \frac{5}{9}(F - 32)$. If the temperature in degrees Celsius ranges over the interval $25 \le C \le 40$ on a certain day, what is the temperature range in degrees Fahrenheit that day?

48. Every integer is either even or odd. The even integers are those that are divisible by 2, so n is even if and only if $n = 2k$ for some integer k. Each odd integer is one unit larger than an even integer, so n is odd if and only if $n = 2k + 1$ for some integer k. Show:
(a) If n is even, then so is n^2
(b) If n is odd, then so is n^2.

49. Prove the following results about sums of rational and irrational numbers:
(a) rational + rational = rational
(b) rational + irrational = irrational.

50. Prove the following results about products of rational and irrational numbers:
(a) rational · rational = rational
(b) rational · irrational = irrational (provided the rational factor is nonzero).

51. Show that the sum or product of two irrational numbers can be rational or irrational.

52. Classify the following as rational or irrational and justify your conclusion.
(a) $3 + \pi$ (b) $\frac{3}{4}\sqrt{2}$
(c) $\sqrt{8}\sqrt{2}$ (d) $\sqrt{\pi}$
(See Exercises 49 and 50.)

53. Prove: The average of two rational numbers is a rational number, but the average of two irrational numbers can be rational or irrational.

54. Can a rational number satisfy $10^x = 3$?

55. Solve: $8x^3 - 4x^2 - 2x + 1 < 0$.

56. Solve: $12x^3 - 20x^2 \ge -11x + 2$.

57. Prove: If a, b, c, and d are positive numbers such that $a < b$ and $c < d$, then $ac < bd$. (This result gives conditions under which inequalities can be "multiplied together.")

58. Is the number represented by the decimal

$$0.101001000100001000001\ldots$$

rational or irrational? Explain your reasoning.

APPENDIX B

Absolute Value

ABSOLUTE VALUE

B.1 DEFINITION. The ***absolute value*** or ***magnitude*** of a real number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example 1

$$|5| = 5 \qquad |-\tfrac{4}{7}| = -(-\tfrac{4}{7}) = \tfrac{4}{7} \qquad |0| = 0$$

Since $5 > 0$ | Since $-\frac{4}{7} < 0$ | Since $0 \geq 0$ ◀

Note that the effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative.

Example 2 Solve $|x - 3| = 4$.

Solution. Depending on whether $x - 3$ is positive or negative, the equation $|x - 3| = 4$ can be written as

$$x - 3 = 4 \quad \text{or} \quad x - 3 = -4$$

Solving these two equations gives $x = 7$ and $x = -1$. ◀

Example 3 Solve $|3x - 2| = |5x + 4|$.

Solution. Because two numbers with the same absolute value are either equal or differ in sign, the given equation will be satisfied if either

$$3x - 2 = 5x + 4 \quad \text{or} \quad 3x - 2 = -(5x + 4)$$

Solving the first equation yields $x = -3$ and solving the second yields $x = -\frac{1}{4}$; thus, the given equation has the solutions $x = -3$ and $x = -\frac{1}{4}$. ◀

RELATIONSHIP BETWEEN SQUARE ROOTS AND ABSOLUTE VALUES

Recall from algebra that a number is called a ***square root*** of a if its square is a. Recall also that every positive real number has two square roots, one positive and one negative; the positive square root is denoted by $\sqrt{a}$ and the negative square root by $-\sqrt{a}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, and the negative square root of 9 is $-\sqrt{9} = -3$.

REMARK. Readers who may have been taught to write $\sqrt{9}$ as ± 3 should stop doing so, since it is incorrect.

It is a common error to replace $\sqrt{a^2}$ by a. Although this is correct when a is nonnegative, it is false for negative a. For example, if $a = -4$, then

$$\sqrt{a^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq a$$

A result that is correct for all a is given in the following theorem.

B.2 THEOREM. *For any real number a,*

$$\sqrt{a^2} = |a|$$

Proof. Since $a^2 = (+a)^2 = (-a)^2$, the numbers $+a$ and $-a$ are square roots of a^2. If $a \geq 0$, then $+a$ is the nonnegative square root of a^2, and if $a < 0$, then $-a$ is the nonnegative square root of a^2. Since $\sqrt{a^2}$ denotes the nonnegative square root of a^2, it follows that

$$\sqrt{a^2} = +a \quad \text{if} \quad a \geq 0$$
$$\sqrt{a^2} = -a \quad \text{if} \quad a < 0$$

That is, $\sqrt{a^2} = |a|$. ∎

PROPERTIES OF ABSOLUTE VALUE

B.3 THEOREM. *If a and b are real numbers, then*

(*a*) $|-a| = |a|$ — A number and its negative have the same absolute value.

(*b*) $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.

(*c*) $|a/b| = |a|/|b|$ — The absolute value of a ratio is the ratio of the absolute values.

We will prove parts (*a*) and (*b*) only.

Proof (a). From Theorem B.2,

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

Proof (b). From Theorem B.2 and a basic property of square roots,

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$$ ∎

REMARK. In part (*c*) of Theorem B.3 we did not explicitly state that $b \neq 0$, but this must be so since division by zero is not allowed. Whenever divisions occur in this text, it will be assumed that the denominator is not zero, even if we do not mention it explicitly.

The result in part (*b*) of Theorem B.3 can be extended to three or more factors. More precisely, for any n real numbers, $a_1, a_2, \ldots, a_n$, it follows that

$$|a_1 a_2 \cdots a_n| = |a_1||a_2| \cdots |a_n| \tag{1}$$

In the special case where $a_1, a_2, \ldots, a_n$ have the same value, a, it follows from (1) that

$$|a^n| = |a|^n \tag{2}$$

GEOMETRIC INTERPRETATION OF ABSOLUTE VALUE

The notion of absolute value arises naturally in distance problems. For example, suppose that A and B are points on a coordinate line that have coordinates a and b, respectively. Depending on the relative positions of the points, the distance d between them will be $b-a$ or $a-b$ (Figure B.1). In either case, the distance can be written as $d = |b-a|$, so we have the following result.

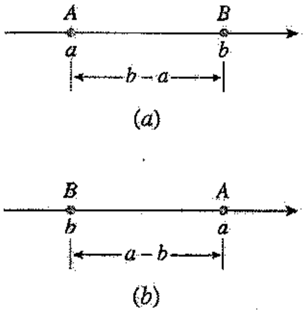


Figure B.1

> **B.4** THEOREM (*Distance Formula*). *If A and B are points on a coordinate line with coordinates a and b, respectively, then the distance d between A and B is* $d = |b-a|$.

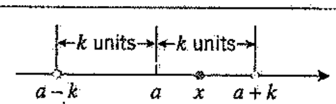
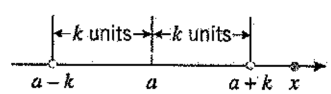
This theorem provides useful geometric interpretations of some common mathematical expressions:

EXPRESSION	GEOMETRIC INTERPRETATION ON A COORDINATE LINE
$\|x-a\|$	The distance between x and a
$\|x+a\|$	The distance between x and $-a$ (since $\|x+a\| = \|x-(-a)\|$)
$\|x\|$	The distance between x and the origin (since $\|x\| = \|x-0\|$)

INEQUALITIES WITH ABSOLUTE VALUES

Inequalities of the form $|x-a| < k$ and $|x-a| > k$ arise so often that we have summarized the key facts about them in Table 1.

Table 1

INEQUALITY ($k > 0$)	GEOMETRIC INTERPRETATION	FIGURE	ALTERNATIVE FORMS OF THE INEQUALITY
$\|x-a\| < k$	x is within k units of a.		$-k < x-a < k$ $a-k < x < a+k$
$\|x-a\| > k$	x is more than k units away from a.		$x-a < -k$ or $x-a > k$ $x < a-k$ or $x > a+k$

REMARK. The statements in this table remain true if $<$ is replaced by $\leq$ and $>$ by $\geq$, and if the open dots are replaced by closed dots in the illustrations.

Example 4 Solve

(a) $|x-3| < 4$ (b) $|x+4| \geq 2$ (c) $\dfrac{1}{|2x-3|} > 5$

Solution (a). The inequality $|x-3| < 4$ can be rewritten as

$$-4 < x-3 < 4$$

Adding 3 throughout yields

$$-1 < x < 7$$

which can be written in interval notation as $(-1, 7)$. Observe that this solution set consists of all x that are within 4 units of 3 on a number line (Figure B.2), which is consistent with Table 1.

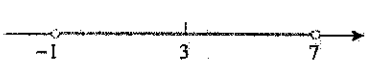


Figure B.2

Solution (b). The inequality $|x+4| \geq 2$ will be satisfied if

$$x+4 \leq -2 \quad \text{or} \quad x+4 \geq 2$$

Solving for x in the two cases yields

$$x \leq -6 \quad \text{or} \quad x \geq -2$$

which can be expressed in interval notation as

$$(-\infty, -6] \cup [-2, +\infty)$$

Observe that the solution set consists of all x that are at least 2 units away from -4 on a number line (Figure B.3), which is consistent with Table 1 and the remark that follows it.

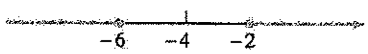


Figure B.3

Solution (c). Observe first that $x = \frac{3}{2}$ results in a division by zero, so this value of x cannot be in the solution set. Putting this aside for the moment, we will begin by taking reciprocals on both sides and reversing the sense of the inequality in accordance with Theorem A.1(e) of Appendix A; then we will use Theorem B.3 to rewrite the inequality $1/|2x-3| > 5$ in a more familiar form:

$$|2x-3| < \tfrac{1}{5}$$

$$|2||x-\tfrac{3}{2}| < \tfrac{1}{5} \qquad \text{Theorem B.3}(b)$$

$$|x-\tfrac{3}{2}| < \tfrac{1}{10} \qquad \text{We multiplied both sides by } 1/|2| = 1/2.$$

$$-\tfrac{1}{10} < x-\tfrac{3}{2} < \tfrac{1}{10} \qquad \text{Table 1}$$

$$\tfrac{7}{5} < x < \tfrac{8}{5} \qquad \text{We added } 3/2 \text{ throughout.}$$

As noted earlier, we must eliminate $x = \frac{3}{2}$ to avoid a division by zero, so the solution set is

$$\tfrac{7}{5} < x < \tfrac{3}{2} \quad \text{or} \quad \tfrac{3}{2} < x < \tfrac{8}{5}$$

which can be expressed in interval notation as $\left(\frac{7}{5}, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \frac{8}{5}\right)$. (See Figure B.4.) ◀

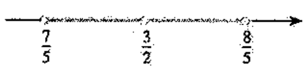


Figure B.4

AN INEQUALITY FROM CALCULUS

One of the most important inequalities in calculus is

$$0 < |x-a| < \delta \tag{3}$$

where δ (Greek "delta") is a positive real number. This is equivalent to the two inequalities

$$0 < |x-a| \quad \text{and} \quad |x-a| < \delta$$

the first of which is satisfied by all x except $x = a$, and the second of which is satisfied by all x that are within δ units of a on a coordinate line. Combining these two restrictions, we conclude that the solution set of (3) consists of all x in the interval $(a-\delta, a+\delta)$ except $x = a$ (Figure B.5). Stated another way, the solution set of (3) is

$$(a-\delta, a) \cup (a, a+\delta) \tag{4}$$

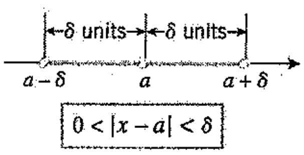


Figure B.5

THE TRIANGLE INEQUALITY

It is *not* generally true that $|a+b| = |a| + |b|$. For example, if $a = 1$ and $b = -1$, then $|a+b| = 0$, whereas $|a| + |b| = 2$. It is true, however, that *the absolute value of a sum is always less than or equal to the sum of the absolute values*. This is the content of the following useful theorem, called the ***triangle inequality***.

> **B.5 THEOREM (*Triangle Inequality*).** *If a and b are any real numbers, then*
>
> $$|a+b| \leq |a| + |b| \tag{5}$$

Proof. Observe first that a satisfies the inequality

$$-|a| \leq a \leq |a|$$

because either $a = |a|$ or $a = -|a|$, depending on the sign of a. The corresponding inequality for b is

$$-|b| \le b \le |b|$$

Adding the two inequalities we obtain

$$-(|a| + |b|) \le a + b \le (|a| + |b|) \tag{6}$$

Let us now consider the cases $a + b \ge 0$ and $a + b < 0$ separately. In the first case, $a + b = |a + b|$, so the right-hand inequality in (6) yields the triangle inequality (5). In the second case, $a + b = -|a + b|$, so the left-hand inequality in (6) can be written as

$$-(|a| + |b|) \le -|a + b|$$

which yields the triangle inequality (5) on multiplying by -1. ∎

REMARK. The name "triangle inequality" arises from a geometric interpretation of the inequality that can be made when a and b are complex numbers. A more detailed explanation is outside the scope of this text.

EXERCISE SET B

1. Compute $|x|$ if
   (a) $x = 7$ (b) $x = -\sqrt{2}$
   (c) $x = k^2$ (d) $x = -k^2$.
2. Rewrite $\sqrt{(x-6)^2}$ without using a square root or absolute value sign.

In Exercises 3–10, find all values of x for which the given statement is true.

3. $|x - 3| = 3 - x$
4. $|x + 2| = x + 2$
5. $|x^2 + 9| = x^2 + 9$
6. $|x^2 + 5x| = x^2 + 5x$
7. $|3x^2 + 2x| = x|3x + 2|$
8. $|6 - 2x| = 2|x - 3|$
9. $\sqrt{(x+5)^2} = x + 5$
10. $\sqrt{(3x-2)^2} = 2 - 3x$
11. Verify $\sqrt{a^2} = |a|$ for $a = 7$ and $a = -7$.
12. Verify the inequalities $-|a| \le a \le |a|$ for $a = 2$ and for $a = -5$.
13. Let A and B be points with coordinates a and b. In each part find the distance between A and B.
    (a) $a = 9,\ b = 7$ (b) $a = 2,\ b = 3$
    (c) $a = -8,\ b = 6$ (d) $a = \sqrt{2},\ b = -3$
    (e) $a = -11,\ b = -4$ (f) $a = 0,\ b = -5$
14. Is the equality $\sqrt{a^4} = a^2$ valid for all values of a? Explain.
15. Let A and B be points with coordinates a and b. In each part, use the given information to find b.
    (a) $a = -3$, B is to the left of A, and $|b - a| = 6$.
    (b) $a = -2$, B is to the right of A, and $|b - a| = 9$.
    (c) $a = 5$, $|b - a| = 7$, and $b > 0$.
16. Let E and F be points with coordinates e and f. In each part, determine whether E is to the left or to the right of F on a coordinate line.
    (a) $f - e = 4$ (b) $e - f = 4$
    (c) $f - e = -6$ (d) $e - f = -7$

In Exercises 17–24, solve for x.

17. $|6x - 2| = 7$
18. $|3 + 2x| = 11$
19. $|6x - 7| = |3 + 2x|$
20. $|4x + 5| = |8x - 3|$
21. $|9x| - 11 = x$
22. $2x - 7 = |x + 1|$
23. $\left|\dfrac{x+5}{2-x}\right| = 6$
24. $\left|\dfrac{x-3}{x+4}\right| = 5$

In Exercises 25–36, solve for x and express the solution in terms of intervals.

25. $|x + 6| < 3$
26. $|7 - x| \le 5$
27. $|2x - 3| \le 6$
28. $|3x + 1| < 4$
29. $|x + 2| > 1$
30. $|\frac{1}{2}x - 1| \ge 2$
31. $|5 - 2x| \ge 4$
32. $|7x + 1| > 3$
33. $\dfrac{1}{|x - 1|} < 2$
34. $\dfrac{1}{|3x + 1|} \ge 5$
35. $\dfrac{3}{|2x - 1|} \ge 4$
36. $\dfrac{2}{|x + 3|} < 1$
37. For which values of x is $\sqrt{(x^2 - 5x + 6)^2} = x^2 - 5x + 6$?
38. Solve $3 \le |x - 2| \le 7$ for x.
39. Solve $|x - 3|^2 - 4|x - 3| = 12$ for x. [*Hint:* Begin by letting $u = |x - 3|$.]
40. Verify the triangle inequality $|a + b| \le |a| + |b|$ (Theorem B.5) for
    (a) $a = 3,\ b = 4$ (b) $a = -2,\ b = 6$
    (c) $a = -7,\ b = -8$ (d) $a = -4,\ b = 4$.
41. Prove: $|a - b| \le |a| + |b|$.
42. Prove: $|a| - |b| \le |a - b|$.
43. Prove: $\big|\, |a| - |b| \,\big| \le |a - b|$. [*Hint:* Use Exercise 42.]

APPENDIX C

Coordinate Planes and Lines

RECTANGULAR COORDINATE SYSTEMS

Just as points on a coordinate line can be associated with real numbers, so points in a plane can be associated with pairs of real numbers by introducing a ***rectangular coordinate system*** (also called a ***Cartesian coordinate system***). A rectangular coordinate system consists of two perpendicular coordinate lines, called ***coordinate axes***, that intersect at their origins. Usually, but not always, one axis is horizontal with its positive direction to the right, and the other is vertical with its positive direction up. The intersection of the axes is called the ***origin*** of the coordinate system.

It is common to call the horizontal axis the ***x-axis*** and the vertical axis the ***y-axis***, in which case the plane and the axes together are referred to as the ***xy-plane*** (Figure C.1). Although labeling the axes with the letters x and y is common, other letters may be more appropriate in specific applications. Figure C.2 shows a uv-plane and a ts-plane—the first letter in the name of the plane always refers to the horizontal axis and the second to the vertical axis.

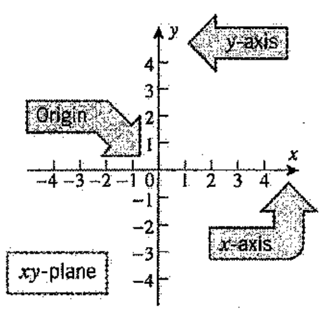


Figure C.1

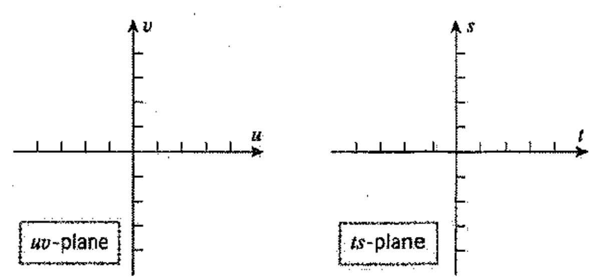


Figure C.2

COORDINATES

Every point P in a coordinate plane can be associated with a unique ordered pair of real numbers by drawing two lines through P, one perpendicular to the x-axis and the other perpendicular to the y-axis (Figure C.3). If the first line intersects the x-axis at the point with coordinate a and the second line intersects the y-axis at the point with coordinate b, then we associate the ordered pair of real numbers (a, b) with the point P. The number a is called the ***x-coordinate*** or ***abscissa*** of P and the number b is called the ***y-coordinate*** or ***ordinate*** of P. We will say that P has ***coordinates*** (a, b) and write $P(a, b)$ when we want to emphasize that the coordinates of P are (a, b). We can also reverse the above procedure and find the point P associated with the coordinates (a, b) by locating the intersection of the dashed

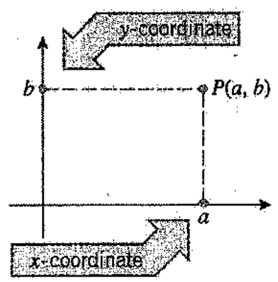


Figure C.3

lines in Figure C.3. Because of this one-to-one correspondence between coordinates and points, we will sometimes blur the distinction between points and ordered pairs of numbers by talking about the *point* (a, b).

REMARK. Recall that the symbol (a, b) also denotes the open interval between a and b; the appropriate interpretation will usually be clear from the context.

In a rectangular coordinate system the coordinate axes divide the rest of the plane into four regions called ***quadrants***. These are numbered counterclockwise with roman numerals as shown in Figure C.4. As indicated in that figure, it is easy to determine the quadrant in which a given point lies from the signs of its coordinates: a point with two positive coordinates $(+, +)$ lies in Quadrant I, a point with a negative x-coordinate and a positive y-coordinate $(-, +)$ lies in Quadrant II, and so forth. Points with a zero x-coordinate lie on the y-axis and points with a zero y-coordinate lie on the x-axis.

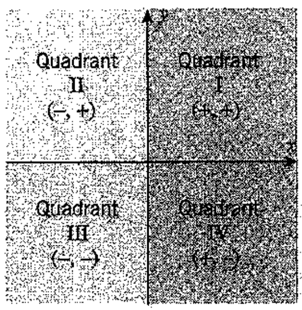


Figure C.4

To ***plot*** a point $P(a, b)$ means to locate the point with coordinates (a, b) in a coordinate plane. For example, in Figure C.5 we have plotted the points

$$P(2, 5), \quad Q(-4, 3), \quad R(-5, -2), \quad \text{and} \quad S(4, -3)$$

Observe how the signs of the coordinates identify the quadrants in which the points lie.

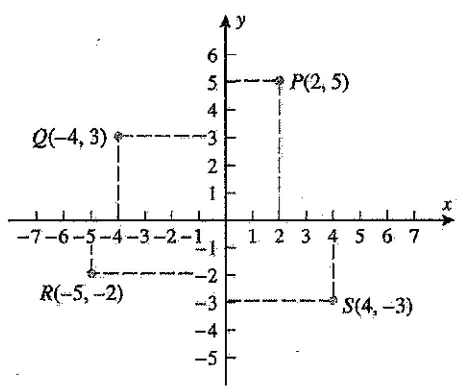


Figure C.5

GRAPHS

The correspondence between points in a plane and ordered pairs of real numbers makes it possible to visualize algebraic equations as geometric curves, and, conversely, to represent geometric curves by algebraic equations. To understand how this is done, suppose that we have an xy-coordinate system and an equation involving two variables x and y, say

$$6x - 4y = 10, \quad y = \sqrt{x}, \quad x = y^3 + 1, \quad \text{or} \quad x^2 + y^2 = 1$$

We define a ***solution*** of such an equation to be any ordered pair of real numbers (a, b) whose coordinates satisfy the equation when we substitute $x = a$ and $y = b$. For example, the ordered pair $(3, 2)$ is a solution of the equation $6x - 4y = 10$, since the equation is satisfied by $x = 3$ and $y = 2$ (verify). However, the ordered pair $(2, 0)$ is not a solution of this equation, since the equation is not satisfied by $x = 2$ and $y = 0$ (verify).

The following definition makes the association between equations in x and y and curves in the xy-plane.

C.1 DEFINITION. The set of all solutions of an equation in x and y is called the ***solution set*** of the equation, and the set of all points in the xy-plane whose coordinates are members of the solution set is called the ***graph*** of the equation.

One of the main themes in calculus is to identify the exact shape of a graph. Point plotting is one approach to obtaining a graph, but this method has limitations, as discussed in the following example.

Example 1 Sketch the graph of $y = x^2$.

Solution. The solution set of the equation has infinitely many members, since we can substitute an arbitrary value for x into the right side of $y = x^2$ and compute the associated y to obtain a point (x, y) in the solution set. The fact that the solution set has infinitely many members means that we cannot obtain the *entire* graph of $y = x^2$ by point plotting. However, we can obtain an *approximation* to the graph by plotting some sample members of the solution set and connecting them with a smooth curve, as in Figure C.6. The problem with this method is that we cannot be sure how the graph behaves *between* the plotted points. For example, the curves in Figure C.7 also pass through the plotted points and hence are legitimate candidates for the graph in the absence of additional information. Moreover, even if we use a graphing calculator or a computer program to generate the graph, as in Figure C.8, we have the same problem because graphing technology uses point-plotting algorithms to generate graphs. Indeed, in Section 1.3 of the text we see examples where graphing technology can be fooled into producing grossly inaccurate graphs. ◀

x	$y = x^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)
−1	1	(−1, 1)
−2	4	(−2, 4)
−3	9	(−3, 9)

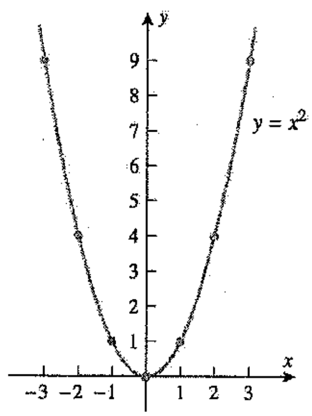


Figure C.6

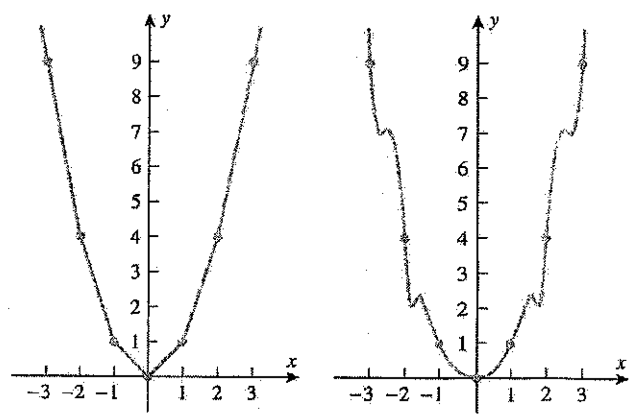
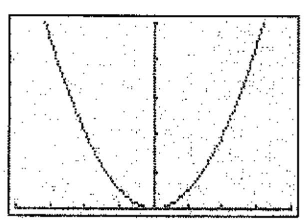


Figure C.7



[−4, 4] × [0, 10]
xScl = 1, yScl = 2

$y = x^2$

Figure C.8

In spite of its limitations, point plotting by hand or with the help of graphing technology can be useful, so here are two more examples.

Example 2 Sketch the graph of $y = \sqrt{x}$.

Solution. If $x < 0$, then $\sqrt{x}$ is an imaginary number. Thus, we can only plot points for which $x \geq 0$, since points in the xy-plane have real coordinates. Figure C.9 shows the graph obtained by point plotting and a graph obtained with a graphing calculator. ◀

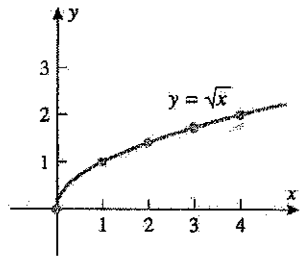
Example 3 Sketch the graph of $y^2 - 2y - x = 0$.

Solution. To calculate coordinates of points on the graph of an equation in x and y, it is desirable to have y expressed in terms of x or x in terms of y. In this case it is easier to express x in terms of y, so we rewrite the equation as

$$x = y^2 - 2y$$

Members of the solution set can be obtained from this equation by substituting arbitrary values for y in the right side and computing the associated values of x (Figure C.10). ◀

x	$y = \sqrt{x}$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	$\sqrt{2}$	$(2, \sqrt{2}) \approx (2, 1.4)$
3	$\sqrt{3}$	$(3, \sqrt{3}) \approx (3, 1.7)$
4	2	(4, 2)



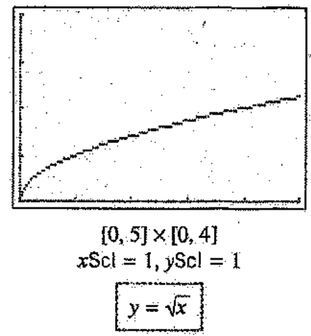
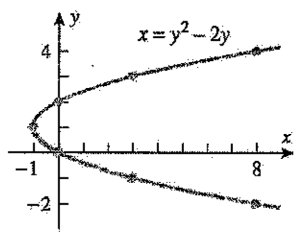


Figure C.9



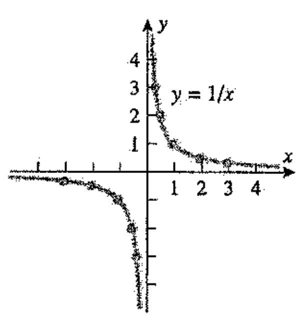
y	$x = y^2 - 2y$	(x, y)
−2	8	(8, −2)
−1	3	(3, −1)
0	0	(0, 0)
1	−1	(−1, 1)
2	0	(0, 2)
3	3	(3, 3)
4	8	(8, 4)

Figure C.10

REMARK. Most graphing calculators and computer graphing programs require that y be expressed in terms of x to generate a graph in the xy-plane. In Section 1.8 we discuss a method for circumventing this restriction.

Example 4 Sketch the graph of $y = 1/x$.

Solution. Because $1/x$ is undefined at $x = 0$, we can only plot points for which $x \neq 0$. This forces a break, called a *discontinuity*, in the graph at $x = 0$ (Figure C.11). ◀



x	$y = 1/x$	(x, y)
$\frac{1}{3}$	3	$(\frac{1}{3}, 3)$
$\frac{1}{2}$	2	$(\frac{1}{2}, 2)$
1	1	(1, 1)
2	$\frac{1}{2}$	$(2, \frac{1}{2})$
3	$\frac{1}{3}$	$(3, \frac{1}{3})$
$-\frac{1}{3}$	−3	$(-\frac{1}{3}, -3)$
$-\frac{1}{2}$	−2	$(-\frac{1}{2}, -2)$
−1	−1	(−1, −1)
−2	$-\frac{1}{2}$	$(-2, -\frac{1}{2})$
−3	$-\frac{1}{3}$	$(-3, -\frac{1}{3})$

Figure C.11

INTERCEPTS

Points where a graph intersects the coordinate axes are of special interest in many problems. As illustrated in Figure C.12, intersections of a graph with the x-axis have the form $(a, 0)$ and intersections with the y-axis have the form $(0, b)$. The number a is called an ***x-intercept*** of the graph and the number b a ***y-intercept***.

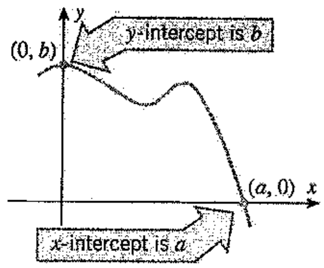


Figure C.12

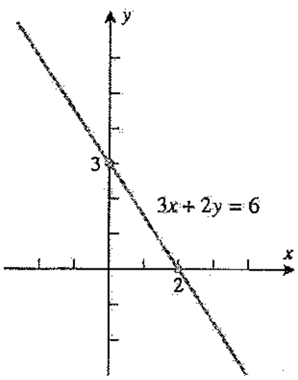


Figure C.13

Example 5 Find all intercepts of

(a) $3x + 2y = 6$ (b) $x = y^2 - 2y$ (c) $y = 1/x$

Solution (a). To find the x-intercepts we set $y = 0$ and solve for x:

$$3x = 6 \quad \text{or} \quad x = 2$$

To find the y-intercepts we set $x = 0$ and solve for y:

$$2y = 6 \quad \text{or} \quad y = 3$$

As we will see later, the graph of $3x + 2y = 6$ is the line shown in Figure C.13.

Solution (b). To find the x-intercepts, set $y = 0$ and solve for x:

$$x = 0$$

Thus, $x = 0$ is the only x-intercept. To find the y-intercepts, set $x = 0$ and solve for y:

$$y^2 - 2y = 0$$

$$y(y - 2) = 0$$

So the y-intercepts are $y = 0$ and $y = 2$. The graph is shown in Figure C.10.

Solution (c). To find the x-intercepts, set $y = 0$:

$$\frac{1}{x} = 0$$

This equation has no solutions (why?), so there are no x-intercepts. To find y-intercepts we would set $x = 0$ and solve for y. But, substituting $x = 0$ leads to a division by zero, which is not allowed, so there are no y-intercepts either. The graph of the equation is shown in Figure C.11. ◀

SLOPE

To obtain equations of lines we will first need to discuss the concept of *slope*, which is a numerical measure of the "steepness" of a line.

Consider a particle moving left to right along a *nonvertical* line from a point $P_1(x_1, y_1)$ to a point $P_2(x_2, y_2)$. As shown in Figure C.14, the particle moves $y_2 - y_1$ units in the y-direction as it travels $x_2 - x_1$ units in the positive x-direction. The vertical change $y_2 - y_1$ is called the ***rise***, and the horizontal change $x_2 - x_1$ the ***run***. The ratio of the rise over the run can be used to measure the steepness of the line, which leads us to the following definition.

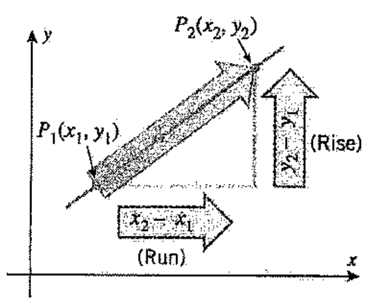


Figure C.14

C.2 DEFINITION. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a nonvertical line, then the ***slope*** m of the line is defined by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1}$$

REMARK. Observe that this definition does not apply to vertical lines. For such lines we have $x_2 = x_1$ (a zero run), which means that the formula for m involves a division by zero. For this reason, the slope of a vertical line is ***undefined***, which is sometimes described informally by stating that a vertical line has ***infinite slope***.

When calculating the slope of a nonvertical line from Formula (1), it does not matter which two points on the line you use for the calculation, as long as they are distinct. This can be proved using Figure C.15 and similar triangles to show that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

Moreover, once you choose two points to use for the calculation, it does not matter which one you call P_1 and which one you call P_2 because reversing the points reverses the sign of both the numerator and denominator of (1) and hence has no effect on the ratio.

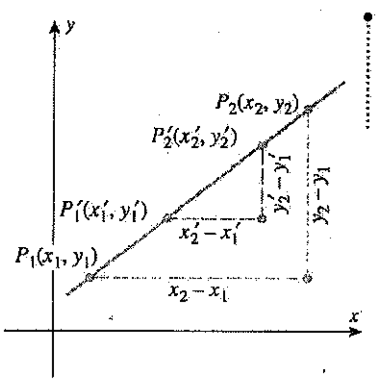


Figure C.15

Example 6 In each part find the slope of the line through

(a) the points (6, 2) and (9, 8)
(b) the points (2, 9) and (4, 3)
(c) the points (−2, 7) and (5, 7).

Solution.

(a) $m = \dfrac{8-2}{9-6} = \dfrac{6}{3} = 2$ (b) $m = \dfrac{3-9}{4-2} = \dfrac{-6}{2} = -3$ (c) $m = \dfrac{7-7}{5-(-2)} = 0$ ◀

Example 7 Figure C.16 shows the three lines determined by the points in Example 6 and explains the significance of their slopes. ◀

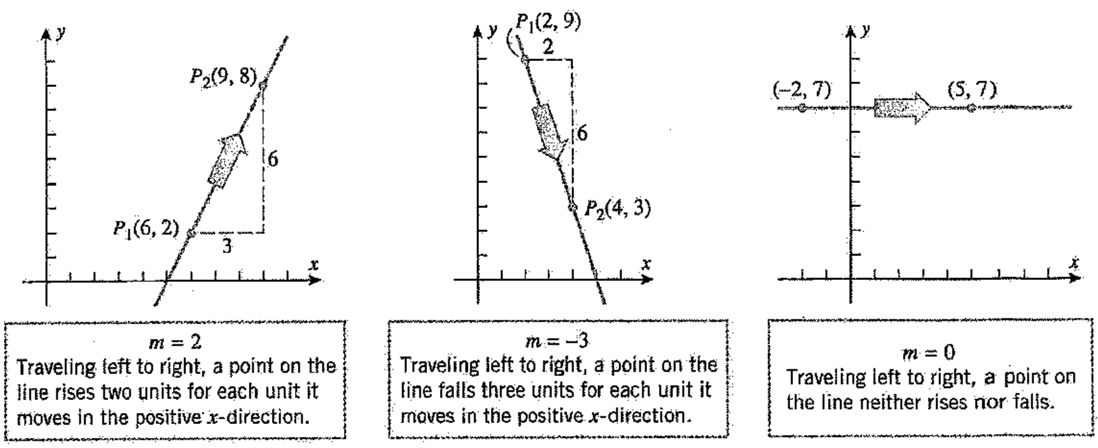


Figure C.16

As illustrated in this example, the slope of a line can be positive, negative, or zero. A positive slope means that the line is inclined upward to the right, a negative slope means that the line is inclined downward to the right, and a zero slope means that the line is horizontal. An undefined slope means that the line is vertical. Figure C.17 shows various lines through the origin with their slopes.

PARALLEL AND PERPENDICULAR LINES

The following theorem shows how slopes can be used to tell whether two lines are parallel or perpendicular.

> **C.3 THEOREM.**
>
> (a) *Two nonvertical lines with slopes m_1 and m_2 are parallel if and only if they have the same slope, that is,*
>
> $$m_1 = m_2$$
>
> (b) *Two nonvertical lines with slopes m_1 and m_2 are perpendicular if and only if the product of their slopes is -1, that is,*
>
> $$m_1 m_2 = -1$$
>
> *This relationship can also be expressed as $m_1 = -1/m_2$ or $m_2 = -1/m_1$, which states that nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of one another.*

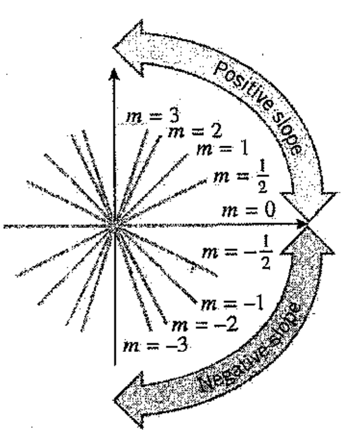


Figure C.17

A complete proof of this theorem is a little tedious, but it is not hard to motivate the results informally. Let us start with part (*a*).

Suppose that L_1 and L_2 are nonvertical parallel lines with slopes m_1 and m_2, respectively. If the lines are parallel to the x-axis, then $m_1 = m_2 = 0$, and we are done. If they are not parallel to the x-axis, then both lines intersect the x-axis; and for simplicity assume that they are oriented as in Figure C.18*a*. On each line choose the point whose run relative to the point of intersection with the x-axis is 1. On line L_1 the corresponding rise will be m_1 and on L_2 it will be m_2. However, because the lines are parallel, the shaded triangles in the figure must be congruent (verify), so $m_1 = m_2$. Conversely, the condition $m_1 = m_2$ can be used to show that the shaded triangles are congruent, from which it follows that the lines make the same angle with the x-axis and hence are parallel (verify).

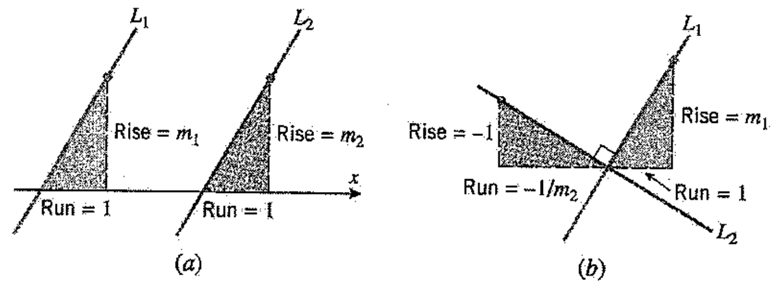


Figure C.18

Now suppose that L_1 and L_2 are nonvertical perpendicular lines with slopes m_1 and m_2, respectively; and for simplicity assume that they are oriented as in Figure C.18*b*. On line L_1 choose the point whose run relative to the point of intersection of the lines is 1, in which case the corresponding rise will be m_1; and on line L_2 choose the point whose rise relative to the point of intersection is -1, in which case the corresponding run will be $-1/m_2$. Because the lines are perpendicular, the shaded triangles in the figure must be congruent (verify), and hence the ratios of corresponding sides of the triangles must be equal. Taking into account that for line L_2 the vertical side of the triangle has length 1 and the horizontal side has length $-1/m_2$ (since m_2 is negative), the congruence of the triangles implies that $m_1/1 = (-1/m_2)/1$ or $m_1 m_2 = -1$. Conversely, the condition $m_1 = -1/m_2$ can be used to show that the shaded triangles are congruent, from which it can be deduced that the lines are perpendicular (verify).

Example 8 Use slopes to show that the points $A(1, 3)$, $B(3, 7)$, and $C(7, 5)$ are vertices of a right triangle.

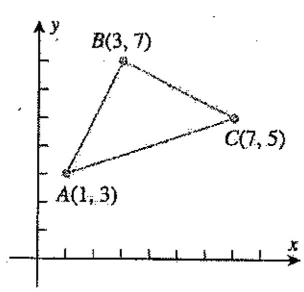


Figure C.19

Solution. We will show that the line through A and B is perpendicular to the line through B and C. The slopes of these lines are

$$\underbrace{m_1 = \frac{7-3}{3-1} = 2}_{\text{Slope of the line through } A \text{ and } B} \quad \text{and} \quad \underbrace{m_2 = \frac{5-7}{7-3} = -\frac{1}{2}}_{\text{Slope of the line through } B \text{ and } C}$$

Since $m_1 m_2 = -1$, the line through A and B is perpendicular to the line through B and C; thus, ABC is a right triangle (Figure C.19). ◀

LINES PARALLEL TO THE COORDINATE AXES

We now turn to the problem of finding equations of lines that satisfy specified conditions. The simplest cases are lines parallel to the coordinate axes. A line parallel to the y-axis intersects the x-axis at some point $(a, 0)$. This line consists precisely of those points whose x-coordinates equal a (Figure C.20). Similarly, a line parallel to the x-axis intersects the y-axis at some point $(0, b)$. This line consists precisely of those points whose y-coordinates equal b (Figure C.20). Thus, we have the following theorem.

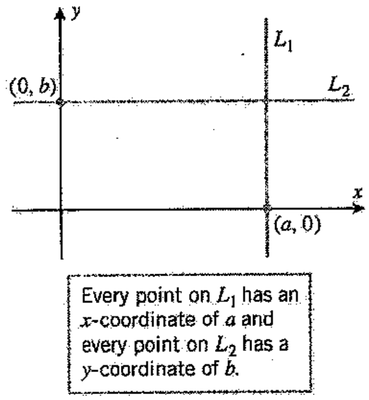


Figure C.20

> C.4 THEOREM. *The vertical line through* $(a, 0)$ *and the horizontal line through* $(0, b)$ *are represented, respectively, by the equations*
>
> $$x = a \quad \text{and} \quad y = b$$

Example 9 The graph of $x = -5$ is the vertical line through $(-5, 0)$, and the graph of $y = 7$ is the horizontal line through $(0, 7)$ (Figure C.21). ◀

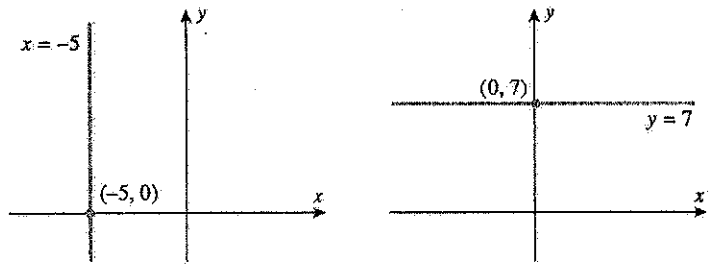


Figure C.21

LINES DETERMINED BY POINT AND SLOPE

There are infinitely many lines that pass through any given point in the plane. However, if we specify the slope of the line in addition to a point on it, then the point and the slope together determine a unique line (Figure C.22).

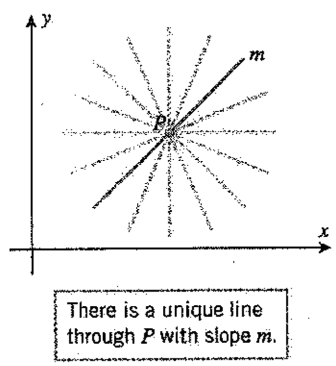


Figure C.22

Let us now consider how to find an equation of a nonvertical line L that passes through a point $P_1(x_1, y_1)$ and has slope m. If $P(x, y)$ is any point on L, different from P_1, then the slope m can be obtained from the points $P(x, y)$ and $P_1(x_1, y_1)$; this gives

$$m = \frac{y - y_1}{x - x_1}$$

which can be rewritten as

$$y - y_1 = m(x - x_1) \tag{2}$$

With the possible exception of (x_1, y_1), we have shown that every point on L satisfies (2). But $x = x_1$, $y = y_1$ satisfies (2), so that all points on L satisfy (2). We leave it as an exercise to show that every point satisfying (2) lies on L.

In summary, we have the following theorem.

> C.5 THEOREM. *The line passing through* $P_1(x_1, y_1)$ *and having slope m is given by the equation*
>
> $$y - y_1 = m(x - x_1) \tag{3}$$
>
> *This is called the* ***point-slope form*** *of the line.*

Example 10 Find the point-slope form of the line through $(4, -3)$ with slope 5.

Solution. Substituting the values $x_1 = 4$, $y_1 = -3$, and $m = 5$ in (3) yields the point-slope form $y + 3 = 5(x - 4)$. ◀

LINES DETERMINED BY SLOPE AND y-INTERCEPT

A nonvertical line crosses the y-axis at some point $(0, b)$. If we use this point in the point-slope form of its equation, we obtain

$$y - b = m(x - 0)$$

which we can rewrite as $y = mx + b$. To summarize:

C.6 THEOREM. *The line with y-intercept b and slope m is given by the equation*

$$y = mx + b \tag{4}$$

This is called the ***slope-intercept form*** *of the line.*

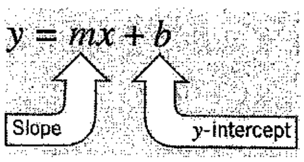


Figure C.23

REMARK. Note that y is alone on one side of Equation (4). When the equation of a line is written in this way the slope of the line and its y-intercept can be determined by inspection of the equation—the slope is the coefficient of x and the y-intercept is the constant term (Figure C.23).

Example 11

EQUATION	SLOPE	y-INTERCEPT
$y = 3x + 7$	$m = 3$	$b = 7$
$y = -x + \frac{1}{2}$	$m = -1$	$b = \frac{1}{2}$
$y = x$	$m = 1$	$b = 0$
$y = \sqrt{2}x - 8$	$m = \sqrt{2}$	$b = -8$
$y = 2$	$m = 0$	$b = 2$

◀

Example 12 Find the slope-intercept form of the equation of the line that satisfies the stated conditions:

(a) slope is -9; crosses the y-axis at $(0, -4)$

(b) slope is 1; passes through the origin

(c) passes through $(5, -1)$; perpendicular to $y = 3x + 4$

(d) passes through $(3, 4)$ and $(2, -5)$.

Solution (a). From the given conditions we have $m = -9$ and $b = -4$, so (4) yields $y = -9x - 4$.

Solution (b). From the given conditions $m = 1$ and the line passes through $(0, 0)$, so $b = 0$. Thus, it follows from (4) that $y = x + 0$ or $y = x$.

Solution (c). The given line has slope 3, so the line to be determined will have slope $m = -\frac{1}{3}$. Substituting this slope and the given point in the point-slope form (3) and then simplifying yields

$$\begin{aligned} y - (-1) &= -\tfrac{1}{3}(x - 5) \\ y &= -\tfrac{1}{3}x + \tfrac{2}{3} \end{aligned}$$

Solution (d). We will first find the point-slope form, then solve for y in terms of x to obtain the slope-intercept form. From the given points the slope of the line is

$$m = \frac{-5 - 4}{2 - 3} = 9$$

We can use either of the given points for (x_1, y_1) in (3). We will use $(3, 4)$. This yields the point-slope form

$$y - 4 = 9(x - 3)$$

Solving for y in terms of x yields the slope-intercept form

$$y = 9x - 23$$

We leave it for the reader to show that the same equation results if $(2, -5)$ rather than $(3, 4)$ is used for (x_1, y_1) in (3). ◀

THE GENERAL EQUATION OF A LINE

An equation that is expressible in the form

$$Ax + By + C = 0 \tag{5}$$

where A, B, and C are constants and A and B are not both zero, is called a ***first-degree equation*** in x and y. For example,

$$4x + 6y - 5 = 0$$

is a first-degree equation in x and y since it has form (5) with

$$A = 4, \quad B = 6, \quad C = -5$$

In fact, all the equations of lines studied in this section are first-degree equations in x and y.

The following theorem states that the first-degree equations in x and y are precisely the equations whose graphs in the xy-plane are straight lines.

> C.7 THEOREM. *Every first-degree equation in x and y has a straight line as its graph and, conversely, every straight line can be represented by a first-degree equation in x and y.*

Because of this theorem, (5) is sometimes called the ***general equation*** of a line or a ***linear equation*** in x and y.

Example 13 Graph the equation $3x - 4y + 12 = 0$.

Solution. Since this is a linear equation in x and y, its graph is a straight line. Thus, to sketch the graph we need only plot any two points on the graph and draw the line through them. It is particularly convenient to plot the points where the line crosses the coordinate axes. These points are $(0, 3)$ and $(-4, 0)$ (verify), so the graph is the line in Figure C.24. ◀

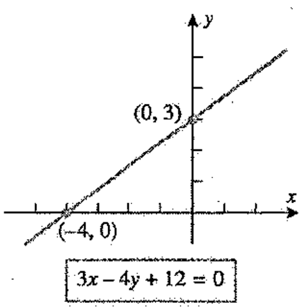


Figure C.24

Example 14 Find the slope of the line in Example 13.

Solution. Solving the equation for y yields

$$y = \tfrac{3}{4}x + 3$$

which is the slope-intercept form of the line. Thus, the slope is $m = \frac{3}{4}$. ◀

EXERCISE SET C

1. Draw the rectangle, three of whose vertices are $(6, 1)$, $(-4, 1)$, and $(6, 7)$, and find the coordinates of the fourth vertex.
2. Draw the triangle whose vertices are $(-3, 2)$, $(5, 2)$, and $(4, 3)$, and find its area.

In Exercises 3 and 4, draw a rectangular coordinate system and sketch the set of points whose coordinates (x, y) satisfy the given conditions.

3. (a) $x = 2$ (b) $y = -3$ (c) $x \geq 0$
   (d) $y = x$ (e) $y \geq x$ (f) $|x| \geq 1$
4. (a) $x = 0$ (b) $y = 0$
   (c) $y < 0$ (d) $x \geq 1$ and $y \leq 2$
   (e) $x = 3$ (f) $|x| = 5$

In Exercises 5–12, sketch the graph of the equation. (A calculating utility will be helpful in some of these problems.)

5. $y = 4 - x^2$
6. $y = 1 + x^2$
7. $y = \sqrt{x - 4}$
8. $y = -\sqrt{x + 1}$
9. $x^2 - x + y = 0$
10. $x = y^3 - y^2$
11. $x^2 y = 2$
12. $xy = -1$

13. Find the slope of the line through
(a) $(-1, 2)$ and $(3, 4)$ (b) $(5, 3)$ and $(7, 1)$
(c) $(4, \sqrt{2})$ and $(-3, \sqrt{2})$ (d) $(-2, -6)$ and $(-2, 12)$.

14. Find the slopes of the sides of the triangle with vertices $(-1, 2)$, $(6, 5)$, and $(2, 7)$.

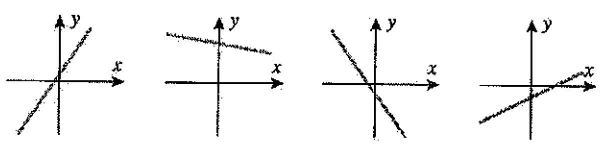
15. Use slopes to determine whether the given points lie on the same line.
(a) $(1, 1)$, $(-2, -5)$, and $(0, -1)$
(b) $(-2, 4)$, $(0, 2)$, and $(1, 5)$

16. Draw the line through $(4, 2)$ with slope
(a) $m = 3$ (b) $m = -2$ (c) $m = -\frac{3}{4}$.

17. Draw the line through $(-1, -2)$ with slope
(a) $m = \frac{3}{5}$ (b) $m = -1$ (c) $m = \sqrt{2}$.

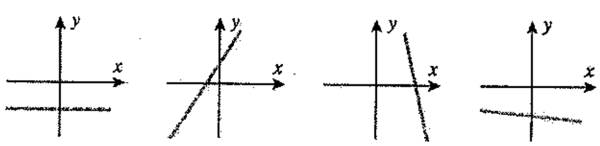
18. An equilateral triangle has one vertex at the origin, another on the x-axis, and the third in the first quadrant. Find the slopes of its sides.

19. List the lines in the accompanying figure in the order of increasing slope.



I II III IV

20. List the lines in the accompanying figure in the order of increasing slope.



I II III IV

21. A particle, initially at $(1, 2)$, moves along a line of slope $m = 3$ to a new position (x, y).
(a) Find y if $x = 5$. (b) Find x if $y = -2$.

22. A particle, initially at $(7, 5)$, moves along a line of slope $m = -2$ to a new position (x, y).
(a) Find y if $x = 9$. (b) Find x if $y = 12$.

23. Let the point $(3, k)$ lie on the line of slope $m = 5$ through $(-2, 4)$; find k.

24. Given that the point $(k, 4)$ is on the line through $(1, 5)$ and $(2, -3)$, find k.

25. Find x if the slope of the line through $(1, 2)$ and $(x, 0)$ is the negative of the slope of the line through $(4, 5)$ and $(x, 0)$.

26. Find x and y if the line through $(0, 0)$ and (x, y) has slope $\frac{1}{2}$, and the line through (x, y) and $(7, 5)$ has slope 2.

27. Use slopes to show that $(3, -1)$, $(6, 4)$, $(-3, 2)$, and $(-6, -3)$ are vertices of a parallelogram.

28. Use slopes to show that $(3, 1)$, $(6, 3)$, and $(2, 9)$ are vertices of a right triangle.

29. Graph the equations
(a) $2x + 5y = 15$ (b) $x = 3$
(c) $y = -2$ (d) $y = 2x - 7$.

30. Graph the equations
(a) $\frac{x}{3} - \frac{y}{4} = 1$ (b) $x = -8$
(c) $y = 0$ (d) $x = 3y + 2$.

31. Graph the equations
(a) $y = 2x - 1$ (b) $y = 3$
(c) $y = -2x$.

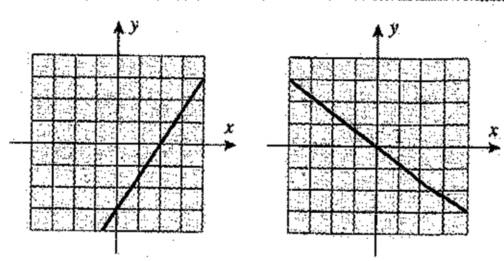
32. Graph the equations
(a) $y = 2 - 3x$ (b) $y = \frac{1}{4}x$
(c) $y = -\sqrt{3}$.

33. Find the slope and y-intercept of
(a) $y = 3x + 2$ (b) $y = 3 - \frac{1}{4}x$
(c) $3x + 5y = 8$ (d) $y = 1$
(e) $\frac{x}{a} + \frac{y}{b} = 1$.

34. Find the slope and y-intercept of
(a) $y = -4x + 2$ (b) $x = 3y + 2$
(c) $\frac{x}{2} + \frac{y}{3} = 1$ (d) $y - 3 = 0$
(e) $a_0 x + a_1 y = 0 \quad (a_1 \neq 0)$.

In Exercises 35 and 36, use the graph to find the equation of the line in slope-intercept form.

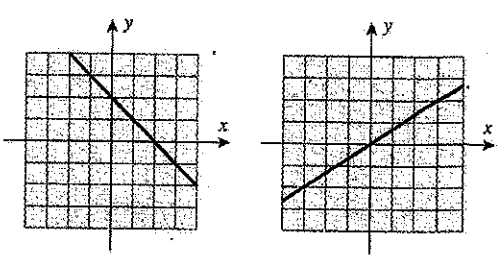
35.



(a) (b)

Figure Ex-35

36.



(a) (b)

Figure Ex-36

In Exercises 37–48, find the slope-intercept form of the line satisfying the given conditions.

37. Slope $= -2$, y-intercept $= 4$.

38. $m = 5, b = -3$.

39. The line is parallel to $y = 4x - 2$ and its y-intercept is 7.

40. The line is parallel to $3x + 2y = 5$ and passes through $(-1, 2)$.

41. The line is perpendicular to $y = 5x + 9$ and its y-intercept is 6.

42. The line is perpendicular to $x - 4y = 7$ and passes through $(3, -4)$.

43. The line passes through $(2, 4)$ and $(1, -7)$.

44. The line passes through $(-3, 6)$ and $(-2, 1)$.

45. The y-intercept is 2 and the x-intercept is -4.

46. The y-intercept is b and the x-intercept is a.

47. The line is perpendicular to the y-axis and passes through $(-4, 1)$.

48. The line is parallel to $y = -5$ and passes through $(-1, -8)$.

49. In each part, classify the lines as parallel, perpendicular, or neither.
(a) $y = 4x - 7$ and $y = 4x + 9$
(b) $y = 2x - 3$ and $y = 7 - \frac{1}{2}x$
(c) $5x - 3y + 6 = 0$ and $10x - 6y + 7 = 0$
(d) $Ax + By + C = 0$ and $Bx - Ay + D = 0$
(e) $y - 2 = 4(x - 3)$ and $y - 7 = \frac{1}{4}(x - 3)$

50. In each part, classify the lines as parallel, perpendicular, or neither.
(a) $y = -5x + 1$ and $y = 3 - 5x$
(b) $y - 1 = 2(x - 3)$ and $y - 4 = -\frac{1}{2}(x + 7)$
(c) $4x + 5y + 7 = 0$ and $5x - 4y + 9 = 0$
(d) $Ax + By + C = 0$ and $Ax + By + D = 0$
(e) $y = \frac{1}{2}x$ and $x = \frac{1}{2}y$

51. For what value of k will the line $3x + ky = 4$
(a) have slope 2
(b) have y-intercept 5
(c) pass through the point $(-2, 4)$
(d) be parallel to the line $2x - 5y = 1$
(e) be perpendicular to the line $4x + 3y = 2$?

52. Sketch the graph of $y^2 = 3x$ and explain how this graph is related to the graphs of $y = \sqrt{3x}$ and $y = -\sqrt{3x}$.

53. Sketch the graph of $(x - y)(x + y) = 0$ and explain how it is related to the graphs of $x - y = 0$ and $x + y = 0$.

54. Graph $F = \frac{9}{5}C + 32$ in a CF-coordinate system.

55. Graph $u = 3v^2$ in a uv-coordinate system.

56. Graph $Y = 4X + 5$ in a YX-coordinate system.

57. A point moves in the xy-plane in such a way that at any time t its coordinates are given by $x = 5t + 2$ and $y = t - 3$. By expressing y in terms of x, show that the point moves along a straight line.

58. A point moves in the xy-plane in such a way that at any time t its coordinates are given by $x = 1 + 3t^2$ and $y = 2 - t^2$. By expressing y in terms of x, show that the point moves along a straight-line path and specify the values of x for which the equation is valid.

59. Find the area of the triangle formed by the coordinate axes and the line through $(1, 4)$ and $(2, 1)$.

60. Draw the graph of $4x^2 - 9y^2 = 0$.

61. In each part, name an appropriate coordinate system for graphing the equation [e.g., an $\alpha\beta$-coordinate system in part (a)], and state whether the graph of the equation is a line in that coordinate system.
(a) $3\alpha - 2\beta = 5$
(b) $A = 2000(1 + 0.06t)$
(c) $A = \pi r^2$
(d) $E = mc^2$ (c constant)
(e) $V = C(1 - rt)$ (r and C constant)
(f) $V = \frac{1}{3}\pi r^2 h$ (r constant)
(g) $V = \frac{1}{3}\pi r^2 h$ (h constant)

APPENDIX D

Distance, Circles, and Quadratic Equations

DISTANCE BETWEEN TWO POINTS IN THE PLANE

Suppose that we are interested in finding the distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the xy-plane. If, as in Figure D.1, we form a right triangle with P_1 and P_2 as vertices, then it follows from Theorem B.4 in Appendix B that the sides of that triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. Thus, it follows from the Theorem of Pythagoras that

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and hence we have the following result.

> **D.1** THEOREM. *The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in a coordinate plane is given by*
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

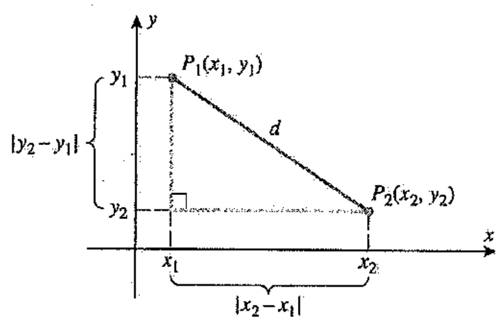


Figure D.1

REMARK. To apply Formula (1) the scales on the coordinate axes must be the same; otherwise, we would not have been able to use the Theorem of Pythagoras in the derivation. Moreover, when using Formula (1) it does not matter which point is labeled P_1 and which one is labeled P_2, since reversing the points changes the signs of $x_2 - x_1$ and $y_2 - y_1$; this has no effect on the value of d because these quantities are squared in the formula. When it is important to emphasize the points, the distance between P_1 and P_2 is denoted by $d(P_1, P_2)$ or $d(P_2, P_1)$.

Example 1 Find the distance between the points $(-2, 3)$ and $(1, 7)$.

Solution. If we let (x_1, y_1) be $(-2, 3)$ and let (x_2, y_2) be $(1, 7)$, then (1) yields

$$d = \sqrt{[1 - (-2)]^2 + [7 - 3]^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$ ◀

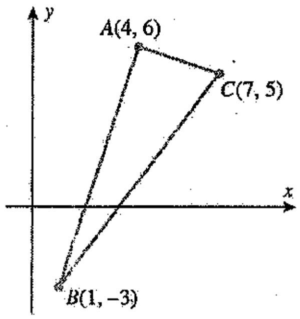


Figure D.2

Example 2 It can be shown that the converse of the Theorem of Pythagoras is true; that is, if the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle. Use this result to show that the points $A(4, 6)$, $B(1, -3)$, and $C(7, 5)$ are vertices of a right triangle.

Solution. The points and the triangle are shown in Figure D.2. From (1), the lengths of the sides of the triangles are

$$d(A, B) = \sqrt{(1-4)^2 + (-3-6)^2} = \sqrt{9+81} = \sqrt{90}$$
$$d(A, C) = \sqrt{(7-4)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$$
$$d(B, C) = \sqrt{(7-1)^2 + [5-(-3)]^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Since

$$[d(A, B)]^2 + [d(A, C)]^2 = [d(B, C)]^2$$

it follows that ΔABC is a right triangle with hypotenuse BC. ◀

THE MIDPOINT FORMULA

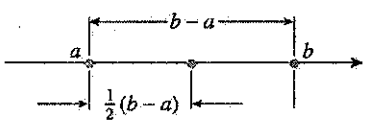


Figure D.3

It is often necessary to find the coordinates of the midpoint of a line segment joining two points in the plane. To derive the midpoint formula, we will start with two points on a coordinate line. If we assume that the points have coordinates a and b and that $a \le b$, then, as shown in Figure D.3, the distance between a and b is $b - a$, and the coordinate of the midpoint between a and b is

$$a + \tfrac{1}{2}(b-a) = \tfrac{1}{2}a + \tfrac{1}{2}b = \tfrac{1}{2}(a+b)$$

which is the arithmetic average of a and b. Had the points been labeled with $b \le a$, the same formula would have resulted (verify). Therefore, *the midpoint of two points on a coordinate line is the arithmetic average of their coordinates, regardless of their relative positions.*

If we now let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points in the plane and $M(x, y)$ the midpoint of the line segment joining them (Figure D.4), then it can be shown using similar triangles that x is the midpoint of x_1 and x_2 on the x-axis and y is the midpoint of y_1 and y_2 on the y-axis, so

$$x = \tfrac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \tfrac{1}{2}(y_1 + y_2)$$

Thus, we have the following result.

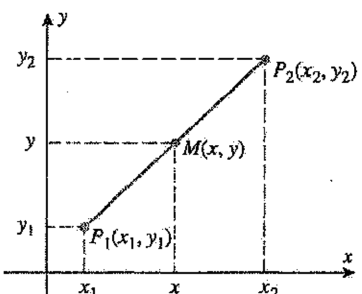


Figure D.4

> **D.2** THEOREM (*The Midpoint Formula*). *The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) in a coordinate plane is*
>
> $$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right) \tag{2}$$

Example 3 Find the midpoint of the line segment joining $(3, -4)$ and $(7, 2)$.

Solution. From (2) the midpoint is

$$\left(\tfrac{1}{2}(3+7), \tfrac{1}{2}(-4+2)\right) = (5, -1)$$ ◀

CIRCLES

If (x_0, y_0) is a fixed point in the plane, then the circle of radius r centered at (x_0, y_0) is the set of all points in the plane whose distance from (x_0, y_0) is r (Figure D.5). Thus, a point (x, y) will lie on this circle if and only if

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = r$$

or equivalently,

$$(x-x_0)^2 + (y-y_0)^2 = r^2 \tag{3}$$

This is called the ***standard form of the equation of a circle.***

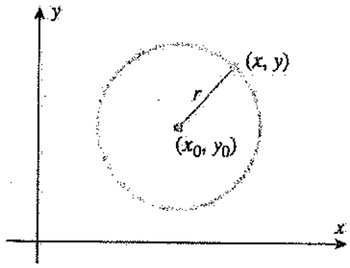


Figure D.5

Example 4 Find an equation for the circle of radius 4 centered at $(-5, 3)$.

Solution. From (3) with $x_0 = -5$, $y_0 = 3$, and $r = 4$ we obtain

$$(x+5)^2 + (y-3)^2 = 16$$

If desired, this equation can be written in an expanded form by squaring the terms and then simplifying:

$$(x^2 + 10x + 25) + (y^2 - 6y + 9) - 16 = 0$$
$$x^2 + y^2 + 10x - 6y + 18 = 0$$

◀

Example 5 Find an equation for the circle with center $(1, -2)$ that passes through $(4, 2)$.

Solution. The radius r of the circle is the distance between $(4, 2)$ and $(1, -2)$, so

$$r = \sqrt{(1-4)^2 + (-2-2)^2} = 5$$

We now know the center and radius, so we can use (3) to obtain the equation

$$(x-1)^2 + (y+2)^2 = 25 \quad \text{or} \quad x^2 + y^2 - 2x + 4y - 20 = 0$$

◀

FINDING THE CENTER AND RADIUS OF A CIRCLE

When you encounter an equation of form (3), you will know immediately that its graph is a circle; its center and radius can then be found from the constants that appear in the equation:

$$\underbrace{(x-x_0)^2}_{x\text{-coordinate of the center is } x_0} + \underbrace{(y-y_0)^2}_{y\text{-coordinate of the center is } y_0} = \underbrace{r^2}_{\text{radius squared}}$$

Example 6

EQUATION OF A CIRCLE	CENTER (x_0, y_0)	RADIUS r
$(x-2)^2 + (y-5)^2 = 9$	$(2, 5)$	3
$(x+7)^2 + (y+1)^2 = 16$	$(-7, -1)$	4
$x^2 + y^2 = 25$	$(0, 0)$	5
$(x-4)^2 + y^2 = 5$	$(4, 0)$	$\sqrt{5}$

◀

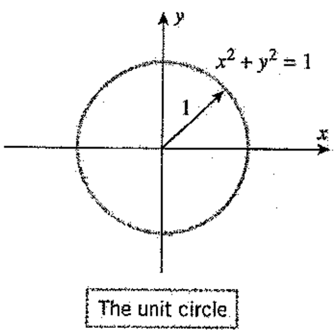


Figure D.6

The circle $x^2 + y^2 = 1$, which is centered at the origin and has radius 1, is of special importance; it is called the ***unit circle*** (Figure D.6).

OTHER FORMS FOR THE EQUATION OF A CIRCLE

An alternative version of Equation (3) can be obtained by squaring the terms and simplifying. This yields an equation of the form

$$x^2 + y^2 + dx + ey + f = 0 \tag{4}$$

where d, e, and f are constants. (See the final equations in Examples 4 and 5.)

Still another version of the equation of a circle can be obtained by multiplying both sides of (4) by a nonzero constant A. This yields an equation of the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \tag{5}$$

where A, D, E, and F are constants and $A \neq 0$.

If the equation of a circle is given by (4) or (5), then the center and radius can be found by first rewriting the equation in standard form, then reading off the center and radius from that equation. The following example shows how to do this using the technique of ***completing the square***. In preparation for the example, recall that completing the square is a method for rewriting an expression of the form

$$x^2 + bx$$

as a difference of two squares. The procedure is to take half the coefficient of x, square it, and then add and subtract that result from the original expression to obtain

$$x^2 + bx = x^2 + bx + (b/2)^2 - (b/2)^2 = [x + (b/2)]^2 - (b/2)^2$$

Example 7 Find the center and radius of the circle with equation

(a) $x^2 + y^2 - 8x + 2y + 8 = 0$ (b) $2x^2 + 2y^2 + 24x - 81 = 0$

Solution (a). First, group the x-terms, group the y-terms, and take the constant to the right side:

$$(x^2 - 8x) + (y^2 + 2y) = -8$$

Next we want to add the appropriate constant within each set of parentheses to complete the square, and subtract the same constant outside the parentheses to maintain equality. The appropriate constant is obtained by taking half the coefficient of the first-degree term and squaring it. This yields

$$(x^2 - 8x + 16) - 16 + (y^2 + 2y + 1) - 1 = -8$$

from which we obtain

$$(x - 4)^2 + (y + 1)^2 = -8 + 16 + 1 \quad \text{or} \quad (x - 4)^2 + (y + 1)^2 = 9$$

Thus from (3), the circle has center $(4, -1)$ and radius 3.

Solution (b). The given equation is of form (5) with $A = 2$. We will first divide through by 2 (the coefficient of the squared terms) to reduce the equation to form (4). Then we will proceed as in part (a) of this example. The computations are as follows:

$$\begin{aligned} x^2 + y^2 + 12x - \tfrac{81}{2} &= 0 && \boxed{\text{We divided through by 2.}} \\ (x^2 + 12x) + y^2 &= \tfrac{81}{2} \\ (x^2 + 12x + 36) + y^2 &= \tfrac{81}{2} + 36 && \boxed{\text{We completed the square.}} \\ (x + 6)^2 + y^2 &= \tfrac{153}{2} \end{aligned}$$

From (3), the circle has center $(-6, 0)$ and radius $\sqrt{\frac{153}{2}}$. ◀

DEGENERATE CASES OF A CIRCLE

There is no guarantee that an equation of form (5) represents a circle. For example, suppose that we divide both sides of (5) by A, then complete the squares to obtain

$$(x - x_0)^2 + (y - y_0)^2 = k$$

Depending on the value of k, the following situations occur:

- $(k > 0)$ The graph is a circle with center (x_0, y_0) and radius $\sqrt{k}$.
- $(k = 0)$ The only solution of the equation is $x = x_0$, $y = y_0$, so the graph is the single point (x_0, y_0).
- $(k < 0)$ The equation has no real solutions and consequently no graph.

Example 8 Describe the graphs of

(a) $(x - 1)^2 + (y + 4)^2 = -9$ (b) $(x - 1)^2 + (y + 4)^2 = 0$

Solution (a). There are no real values of x and y that will make the left side of the equation negative. Thus, the solution set of the equation is empty, and the equation has no graph.

Solution (b). The only values of x and y that will make the left side of the equation 0 are $x = 1$, $y = -4$. Thus, the graph of the equation is the single point $(1, -4)$. ◀

The following theorem summarizes our observations.

> **D.3** THEOREM. *An equation of the form*
>
> $$Ax^2 + Ay^2 + Dx + Ey + F = 0 \tag{6}$$
>
> *where $A \neq 0$, represents a circle, or a point, or else has no graph.*

REMARK. The last two cases in Theorem D.3 are called ***degenerate cases***. In spite of the fact that these degenerate cases can occur, (6) is often called the ***general equation of a circle***.

THE GRAPH of $y = ax^2 + bx + c$

An equation of the form

$$y = ax^2 + bx + c \quad (a \neq 0) \tag{7}$$

is called a ***quadratic equation in x***. Depending on whether a is positive or negative, the graph, which is called a ***parabola***, has one of the two forms shown in Figure D.7. In both cases the parabola is symmetric about a vertical line parallel to the y-axis. This line of symmetry cuts the parabola at a point called the ***vertex***. The vertex is the low point on the curve if $a > 0$ and the high point if $a < 0$.

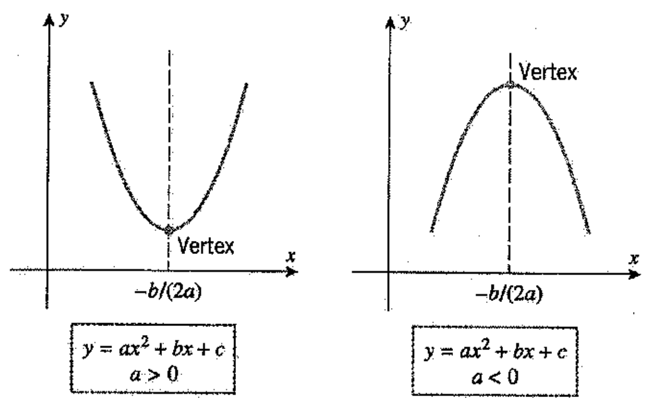


Figure D.7

x	$y = x^2 - 2x - 2$
−1	1
0	−2
1	−3
2	−2
3	1

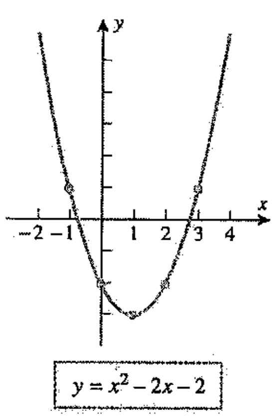


Figure D.8

In the exercises (Exercise 78) we will help the reader show that the x-coordinate of the vertex is given by the formula

$$x = -\frac{b}{2a} \tag{8}$$

With the aid of this formula, a reasonably accurate graph of a quadratic equation in x can be obtained by plotting the vertex and two points on each side of it.

Example 9 Sketch the graph of

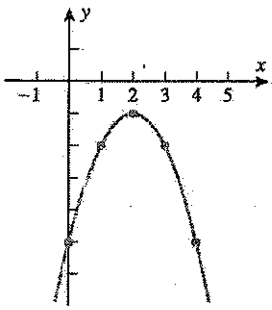
(a) $y = x^2 - 2x - 2$ (b) $y = -x^2 + 4x - 5$

Solution (a). The equation is of form (7) with $a = 1$, $b = -2$, and $c = -2$, so by (8) the x-coordinate of the vertex is

$$x = -\frac{b}{2a} = 1$$

Using this value and two additional values on each side, we obtain Figure D.8.

x	$y = -x^2 + 4x - 5$
0	−5
1	−2
2	−1
3	−2
4	−5



$y = -x^2 + 4x - 5$

Figure D.9

Solution (b). The equation is of form (7) with $a = -1$, $b = 4$, and $c = -5$, so by (8) the x-coordinate of the vertex is

$$x = -\frac{b}{2a} = 2$$

Using this value and two additional values on each side, we obtain the table and graph in Figure D.9. ◀

Quite often the intercepts of a parabola $y = ax^2 + bx + c$ are important to know. The y-intercept, $y = c$, results immediately by setting $x = 0$. However, in order to obtain the x-intercepts, if any, we must set $y = 0$ and then solve the resulting quadratic equation $ax^2 + bx + c = 0$.

Example 10 Solve the inequality

$$x^2 - 2x - 2 > 0$$

Solution. Because the left side of the inequality does not have readily discernible factors, the test-value method illustrated in Example 4 of Appendix A is not convenient to use. Instead, we will give a graphical solution. The given inequality is satisfied for those values of x where the graph of $y = x^2 - 2x - 2$ is above the x-axis. From Figure D.8 those are the values of x to the left of the smaller intercept or to the right of the larger intercept. To find these intercepts we set $y = 0$ to obtain

$$x^2 - 2x - 2 = 0$$

Solving by the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

Thus, the x-intercepts are

$$x = 1 + \sqrt{3} \approx 2.7 \quad \text{and} \quad x = 1 - \sqrt{3} \approx -0.7$$

and the solution set of the inequality is

$$(-\infty, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, +\infty)$$ ◀

REMARK. Note that the decimal approximations of the intercepts calculated in the preceding example agree with the graph in Figure D.8. Observe, however, that we used the exact values of the intercepts to express the solution. The choice of exact versus approximate values is often a matter of judgment that depends on the purpose for which the values are to be used. Numerical approximations often provide a sense of size that exact values do not, but they can introduce severe errors if not used with care.

Example 11 From Figure D.9 we see that the parabola $y = -x^2 + 4x - 5$ has no x-intercepts. This can also be seen algebraically by solving for the x-intercepts. Setting $y = 0$ and solving the resulting equation

$$-x^2 + 4x - 5 = 0$$

by the quadratic formula yields

$$y = \frac{-4 \pm \sqrt{16 - 20}}{-2} = 2 \pm i$$

Because the solutions are not real numbers, there are no x-intercepts. ◀

Example 12 A ball is thrown straight up from the surface of the Earth at time $t = 0$ s with an initial velocity of 24.5 m/s. If air resistance is ignored, it can be shown that the

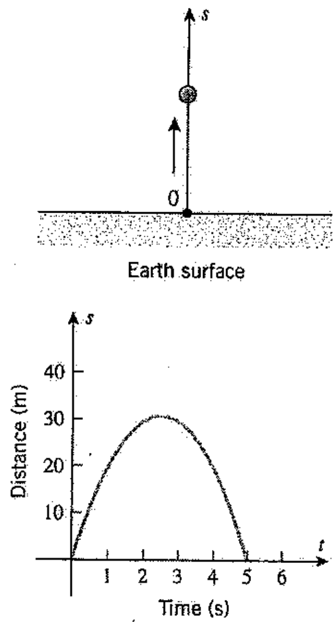


Figure D.10

distance s (in meters) of the ball above the ground after t seconds is given by

$$s = 24.5t - 4.9t^2 \tag{9}$$

(a) Graph s versus t, making the t-axis horizontal and the s-axis vertical.
(b) How high does the ball rise above the ground?

Solution (a). Equation (9) is of form (7) with $a = -4.9$, $b = 24.5$, and $c = 0$, so by (8) the t-coordinate of the vertex is

$$t = -\frac{b}{2a} = -\frac{24.5}{2(-4.9)} = 2.5 \text{ s}$$

and consequently the s-coordinate of the vertex is

$$s = 24.5(2.5) - 4.9(2.5)^2 = 30.625 \text{ m}$$

The factored form of (9) is

$$s = 4.9t(5 - t)$$

so the graph has t-intercepts $t = 0$ and $t = 5$. From the vertex and the intercepts we obtain the graph shown in Figure D.10.

Solution (b). From the s-coordinate of the vertex we deduce that the ball rises 30.625 m above the ground. ◀

THE GRAPH of $x = ay^2 + by + c$

If x and y are interchanged in (7), the resulting equation,

$$x = ay^2 + by + c$$

is called a ***quadratic equation in y***. The graph of such an equation is a parabola with its line of symmetry parallel to the x-axis and its vertex at the point with y-coordinate $y = -b/(2a)$ (Figure D.11). Some problems relating to such equations appear in the exercises.

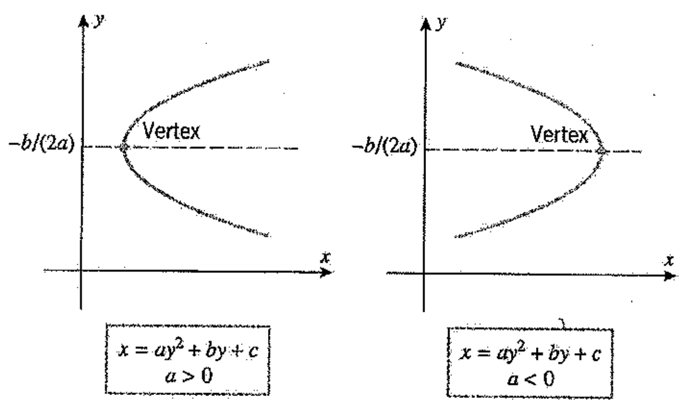


Figure D.11

Exercise Set D

1. Where in this section did we use the fact that the same scale was used on both coordinate axes?

In Exercises 2–5, find
(a) the distance between A and B
(b) the midpoint of the line segment joining A and B.

2. $A(2, 5)$, $B(-1, 1)$
3. $A(7, 1)$, $B(1, 9)$
4. $A(2, 0)$, $B(-3, 6)$
5. $A(-2, -6)$, $B(-7, -4)$

In Exercises 6–10, use the distance formula to solve the given problem.

6. Prove that $(1, 1)$, $(-2, -8)$, and $(4, 10)$ lie on a straight line.
7. Prove that the triangle with vertices $(5, -2)$, $(6, 5)$, $(2, 2)$ is isosceles.

8. Prove that $(1, 3)$, $(4, 2)$, and $(-2, -6)$ are vertices of a right triangle and then specify the vertex at which the right angle occurs.

9. Prove that $(0, -2)$, $(-4, 8)$, and $(3, 1)$ lie on a circle with center $(-2, 3)$.

10. Prove that for all values of t the point $(t, 2t-6)$ is equidistant from $(0, 4)$ and $(8, 0)$.

11. Find k, given that $(2, k)$ is equidistant from $(3, 7)$ and $(9, 1)$.

12. Find x and y if $(4, -5)$ is the midpoint of the line segment joining $(-3, 2)$ and (x, y).

In Exercises 13 and 14, find an equation of the given line.

13. The line is the perpendicular bisector of the line segment joining $(2, 8)$ and $(-4, 6)$.

14. The line is the perpendicular bisector of the line segment joining $(5, -1)$ and $(4, 8)$.

15. Find the point on the line $4x - 2y + 3 = 0$ that is equidistant from $(3, 3)$ and $(7, -3)$. [*Hint:* First find an equation of the line that is the perpendicular bisector of the line segment joining $(3, 3)$ and $(7, -3)$.]

16. Find the distance from the point $(3, -2)$ to the line
(a) $y = 4$ (b) $x = -1$.

17. Find the distance from $(2, 1)$ to the line $4x - 3y + 10 = 0$. [*Hint:* Find the foot of the perpendicular dropped from the point to the line.]

18. Find the distance from $(8, 4)$ to the line $5x + 12y - 36 = 0$. [*Hint:* See the hint in Exercise 17.]

19. Use the method described in Exercise 17 to prove that the distance d from (x_0, y_0) to the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

20. Use the formula in Exercise 19 to solve Exercise 17.

21. Use the formula in Exercise 19 to solve Exercise 18.

22. Prove: For any triangle, the perpendicular bisectors of the sides meet at a point. [*Hint:* Position the triangle with one vertex on the y-axis and the opposite side on the x-axis, so that the vertices are $(0, a)$, $(b, 0)$, and $(c, 0)$.]

In Exercises 23 and 24, find the center and radius of each circle.

23. (a) $x^2 + y^2 = 25$
(b) $(x-1)^2 + (y-4)^2 = 16$
(c) $(x+1)^2 + (y+3)^2 = 5$
(d) $x^2 + (y+2)^2 = 1$

24. (a) $x^2 + y^2 = 9$
(b) $(x-3)^2 + (y-5)^2 = 36$
(c) $(x+4)^2 + (y+1)^2 = 8$
(d) $(x+1)^2 + y^2 = 1$

In Exercises 25–32, find the standard equation of the circle satisfying the given conditions.

25. Center $(3, -2)$; radius $= 4$.

26. Center $(1, 0)$; diameter $= \sqrt{8}$.

27. Center $(-4, 8)$; circle is tangent to the x-axis.

28. Center $(5, 8)$; circle is tangent to the y-axis.

29. Center $(-3, -4)$; circle passes through the origin.

30. Center $(4, -5)$; circle passes through $(1, 3)$.

31. A diameter has endpoints $(2, 0)$ and $(0, 2)$.

32. A diameter has endpoints $(6, 1)$ and $(-2, 3)$.

In Exercises 33–44, determine whether the equation represents a circle, a point, or no graph. If the equation represents a circle, find the center and radius.

33. $x^2 + y^2 - 2x - 4y - 11 = 0$

34. $x^2 + y^2 + 8x + 8 = 0$

35. $2x^2 + 2y^2 + 4x - 4y = 0$

36. $6x^2 + 6y^2 - 6x + 6y = 3$

37. $x^2 + y^2 + 2x + 2y + 2 = 0$

38. $x^2 + y^2 - 4x - 6y + 13 = 0$

39. $9x^2 + 9y^2 = 1$

40. $(x^2/4) + (y^2/4) = 1$

41. $x^2 + y^2 + 10y + 26 = 0$

42. $x^2 + y^2 - 10x - 2y + 29 = 0$

43. $16x^2 + 16y^2 + 40x + 16y - 7 = 0$

44. $4x^2 + 4y^2 - 16x - 24y = 9$

45. Find an equation of
(a) the bottom half of the circle $x^2 + y^2 = 16$
(b) the top half of the circle $x^2 + y^2 + 2x - 4y + 1 = 0$.

46. Find an equation of
(a) the right half of the circle $x^2 + y^2 = 9$
(b) the left half of the circle $x^2 + y^2 - 4x + 3 = 0$.

47. Graph
(a) $y = \sqrt{25 - x^2}$ (b) $y = \sqrt{5 + 4x - x^2}$.

48. Graph
(a) $x = -\sqrt{4 - y^2}$ (b) $x = 3 + \sqrt{4 - y^2}$.

49. Find an equation of the line that is tangent to the circle

$$x^2 + y^2 = 25$$

at the point $(3, 4)$ on the circle.

50. Find an equation of the line that is tangent to the circle at the point P on the circle
(a) $x^2 + y^2 + 2x = 9$; $P(2, -1)$
(b) $x^2 + y^2 - 6x + 4y = 13$; $P(4, 3)$.

51. For the circle $x^2 + y^2 = 20$ and the point $P(-1, 2)$:
(a) Is P inside, outside, or on the circle?

(b) Find the largest and smallest distances between P and points on the circle.

52. Follow the directions of Exercise 51 for the circle

$$x^2 + y^2 - 2y - 4 = 0$$

and the point $P\left(3, \frac{5}{2}\right)$.

53. Referring to the accompanying figure, find the coordinates of the points T and T', where the lines L and L' are tangent to the circle of radius 1 with center at the origin.

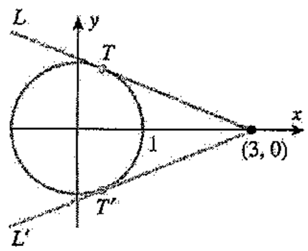


Figure Ex-53

54. A point (x, y) moves so that its distance to $(2, 0)$ is $\sqrt{2}$ times its distance to $(0, 1)$.
(a) Show that the point moves along a circle.
(b) Find the center and radius.

55. A point (x, y) moves so that the sum of the squares of its distances from $(4, 1)$ and $(2, -5)$ is 45.
(a) Show that the point moves along a circle.
(b) Find the center and radius.

56. Find all values of c for which the system of equations

$$\begin{cases} x^2 - y^2 = 0 \\ (x - c)^2 + y^2 = 1 \end{cases}$$

has 0, 1, 2, 3, or 4 solutions. [*Hint:* Sketch a graph.]

In Exercises 57–70, graph the parabola and label the coordinates of the vertex and the intersections with the coordinate axes.

57. $y = x^2 + 2$
58. $y = x^2 - 3$
59. $y = x^2 + 2x - 3$
60. $y = x^2 - 3x - 4$
61. $y = -x^2 + 4x + 5$
62. $y = -x^2 + x$
63. $y = (x - 2)^2$
64. $y = (3 + x)^2$
65. $x^2 - 2x + y = 0$
66. $x^2 + 8x + 8y = 0$
67. $y = 3x^2 - 2x + 1$
68. $y = x^2 + x + 2$
69. $x = -y^2 + 2y + 2$
70. $x = y^2 - 4y + 5$

71. Find an equation of
(a) the right half of the parabola $y = 3 - x^2$
(b) the left half of the parabola $y = x^2 - 2x$.

72. Find an equation of
(a) the upper half of the parabola $x = y^2 - 5$
(b) the lower half of the parabola $x = y^2 - y - 2$.

73. Graph
(a) $y = \sqrt{x + 5}$ (b) $x = -\sqrt{4 - y}$.

74. Graph
(a) $y = 1 + \sqrt{4 - x}$ (b) $x = 3 + \sqrt{y}$.

75. If a ball is thrown straight up with an initial velocity of 32 ft/s, then after t seconds the distance s above its starting height, in feet, is given by $s = 32t - 16t^2$.
(a) Graph this equation in a ts-coordinate system (t-axis horizontal).
(b) At what time t will the ball be at its highest point, and how high will it rise?

76. A rectangular field is to be enclosed with 500 ft of fencing along three sides and by a straight stream on the fourth side. Let x be the length of each side perpendicular to the stream, and let y be the length of the side parallel to the stream.
(a) Express y in terms of x.
(b) Express the area A of the field in terms of x.
(c) What is the largest area that can be enclosed?

77. A rectangular plot of land is to be enclosed using two kinds of fencing. Two opposite sides will have heavy-duty fencing costing \$3/ft, and the other two sides will have standard fencing costing \$2/ft. A total of \$600 is available for the fencing. Let x be the length of each side with the heavy-duty fencing, and let y be the length of each side with the standard fencing.
(a) Express y in terms of x.
(b) Find a formula for the area A of the rectangular plot in terms of x.
(c) What is the largest area that can be enclosed?

78. (a) By completing the square, show that the quadratic equation $y = ax^2 + bx + c$ can be rewritten as

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

if $a \neq 0$.
(b) Use the result in part (a) to show that the graph of the quadratic equation $y = ax^2 + bx + c$ has its high point at $x = -b/(2a)$ if $a < 0$ and its low point there if $a > 0$.

In Exercises 79 and 80, solve the given inequality.

79. (a) $2x^2 + 5x - 1 < 0$ (b) $x^2 - 2x + 3 > 0$

80. (a) $x^2 + x - 1 > 0$ (b) $x^2 - 4x + 6 < 0$

81. At time $t = 0$ a ball is thrown straight up from a height of 5 ft above the ground. After t seconds its distance s, in feet, above the ground is given by $s = 5 + 40t - 16t^2$.
(a) Find the maximum height of the ball above the ground.
(b) Find, to the nearest tenth of a second, the time when the ball strikes the ground.
(c) Find, to the nearest tenth of a second, how long the ball will be more than 12 ft above the ground.

82. Find all values of x at which points on the parabola $y = x^2$ lie below the line $y = x + 3$.